Problem Set 7

(1) $H_{N\times M}$, $N \geq M$

$$H^H H = L L^H$$

$$H = Q \begin{bmatrix} R \\ 0 \end{bmatrix} \quad Q_{N\times N} \quad R_{M\times M}$$

(a) $$H^H H = \left(Q \begin{bmatrix} R \\ 0 \end{bmatrix}\right)^H Q \begin{bmatrix} R \\ 0 \end{bmatrix} = [R^H \ 0] Q^H Q \begin{bmatrix} R \\ 0 \end{bmatrix} = R^H R.$$

$\Rightarrow L = R^H$

(b) $$\begin{bmatrix} \underline{z}_1 \\ \underline{z}_2 \end{bmatrix} = Q^H \underline{x} \qquad \underline{z}_{1\, M\times 1} \qquad \underline{z}_{2\, (N-M)\times 1}.$$

$$\|\underline{x} - H\underline{\omega}\|^2 = \|Q^H(\underline{x} - H\underline{\omega})\|^2 = \|Q^H\underline{x} - Q^H H \underline{\omega}\|^2$$

$$= \left\| \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} - \begin{bmatrix} R \\ 0 \end{bmatrix} \underline{\omega} \right\|^2$$

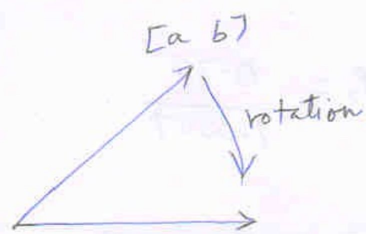
$$= \|\underline{z}_1 - R\underline{\omega}\|^2 + \|\underline{z}_2\|^2.$$

$\Rightarrow$ $\hat{\underline{\omega}}$ that minimizes $\|\underline{x} - H\underline{\omega}\|^2$ can be obtained by finding $\hat{\underline{\omega}}$ that minimizes $\|\underline{z}_1 - R\underline{\omega}\|^2$. Note $\|\underline{z}_1 - R\underline{\omega}\|^2 \geq 0$.

$\|\underline{z}_1 - R\underline{\omega}\|^2$ can be made 0 by solving $\underline{z}_1 = \underbrace{R}_{\Delta^{lar} \text{ with the diag.}}\underline{\omega}$
$\Rightarrow$ soln. exists.

This also means that $\min_{\underline{\omega}} \|\underline{x} - H\underline{\omega}\|^2 = \|\underline{z}_2\|^2.$

(2)



$a = 0$ $\quad [0 \ \ b]\theta = \pm[b \ \ 0]$

for $\theta = \begin{bmatrix} 0 & \pm 1 \\ 1 & 0 \end{bmatrix}$

$\underline{a \neq 0.}$

$$[a \ \ b]\theta = [a \ \ b] \begin{bmatrix} 1 & -\rho \\ \rho^* & 1 \end{bmatrix} \frac{1}{\sqrt{1+|\rho|^2}}$$

$$= [a + b\rho^* \ \ \ -a\rho + b] \frac{1}{\sqrt{1+|\rho|^2}}$$

$\rho = b/a \Rightarrow -a\rho + b = 0$

$$a + b\rho^* = a + b\frac{b^*}{a^*}$$

$$= \frac{aa^* + bb^*}{a^*} = \frac{|a|^2 + |b|^2}{a^*}.$$

$$[a\ \ b]\Theta = \left[\frac{|a|^2+|b|^2}{a^*}\quad 0\right]\frac{1}{\sqrt{1+|\rho|^2}}$$

$$= \frac{1}{\sqrt{1+\left|\frac{b}{a}\right|^2}}\frac{(|a|^2+|b|^2)}{a^*}[1\quad 0]$$

$$= \frac{\pm|a|\sqrt{|a|^2+|b|^2}}{a^*}[1\quad 0] = \pm e^{j\phi_a}\sqrt{|a|^2+|b|^2}\,[1\quad 0]$$

(3)

$$A = \begin{bmatrix} 1 & 0.75 & 0.25 \\ 0.4 & 0.2 & 0.2 \end{bmatrix}$$

Take (1,1) & (1,3) entry and define

$$\theta_1 = \frac{1}{\sqrt{1+\rho_1^2}}\begin{bmatrix} 1 & -\rho_1 \\ \rho_1 & 1 \end{bmatrix} \quad \text{where } \rho_1 = \frac{0.25}{1} = 0.25.$$

$$= \begin{bmatrix} 0.9701 & -0.2425 \\ 0.2425 & 0.9701 \end{bmatrix}$$

$$A\overbrace{\begin{bmatrix} 0.9701 & 0 & -0.2425 \\ 0 & 1 & 0 \\ 0.2425 & 0 & 0.9701 \end{bmatrix}}^{\theta_1'} = \begin{bmatrix} 1.0307 & 0.75 & 0 \\ 0.4365 & 0.2 & 0.097 \end{bmatrix}$$

zero here

Choose (1,1) & (1,2) entry now to define

$$\theta_2 = \frac{1}{\sqrt{1+\rho_2^2}}\begin{bmatrix} 1 & -\rho_2 \\ \rho_2 & 1 \end{bmatrix} \quad \text{where } \rho_2 = \frac{0.75}{1.0307}.$$

$$A\theta_1'\theta_2' = A\theta_1'\left[\begin{array}{c|c} \theta_2 & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & 1 \end{array}\right] = \begin{bmatrix} 1.2748 & 0 & 0 \\ 0.4707 & -0.0951 & 0.0970 \end{bmatrix}$$

Now, using (2,2) & (2,3) entries define

$$\theta_3 = \frac{1}{\sqrt{1+p_3^2}}\begin{bmatrix} 1 & -p_3 \\ p_3 & 1 \end{bmatrix} \quad \text{where } p_3 = \frac{0.097}{-0.0951}$$

$$A\theta_1'\theta_2'\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$-\theta_3$}} \\ 0 & & \end{array}\right] = \begin{bmatrix} 1.2748 & 0 & 0 \\ 0.4707 & 0.1359 & 0 \end{bmatrix} \ldots$$

if θ_3 is used instead of $-\theta_3$, we will get -0.1359 here