Problem Set 5

(1) $\underline{k}_i = \underline{k}_{i-1} + \mu \underline{y}_i g(E_i)$ $\quad \underline{k}_{-1}$ (initial condition) $\rightarrow$ Adaptive filter.

$$X_i = \underline{k}_{o,i}^H \underline{y}_i e^{-j\Omega i} + V_i \quad \text{(Data model)}$$

$$\left.\begin{aligned} \underline{k}_{o,i} &= \underline{k}_o + \underline{\theta}_i \\ \underline{\theta}_i &= \alpha \underline{\theta}_{i-1} + \underline{q}_i \end{aligned}\right\} \quad \underline{\theta}_{-1} \text{ (initial condition)}$$

Let $\tilde{\underline{k}}_i = \underline{k}_{o,i} e^{j\Omega i} - \underline{k}_i$

$$E_{ai} = [\underline{k}_{o,i} e^{j\Omega i} - \underline{k}_{i-1}]^H \underline{y}_i$$

$$E_{pi} = [\underline{k}_{o,i} e^{j\Omega i} - \underline{k}_i]^H \underline{y}_i$$

(a) $\tilde{\underline{k}}_i = \underline{k}_{o,i} e^{j\Omega i} - \underline{k}_i$

$$= \underline{k}_{o,i} e^{j\Omega i} - \underline{k}_{i-1} - \mu \underline{y}_i g(E_i)$$

$$\tilde{\underline{k}}_{i-1} = \underline{k}_{o,i-1} e^{j\Omega(i-1)} - \underline{k}_{i-1}$$

$$\Rightarrow \tilde{\underline{k}}_i = \tilde{\underline{k}}_{i-1} + \underline{k}_{o,i} e^{j\Omega i} - \underline{k}_{o,i-1} e^{j\Omega(i-1)} - \mu \underline{y}_i g(E_i)$$

$$= \tilde{\underline{k}}_{i-1} - \mu \underline{y}_i g(E_i) + e^{j\Omega(i-1)} \underbrace{\left[\underline{k}_{o,i} e^{j\Omega} - \underline{k}_{o,i-1}\right]}_{\triangleq\, \underline{c}_i}$$

$$\underline{c}_i = (\underline{k}_o + \underline{\theta}_i) e^{j\Omega} - (\underline{k}_o + \underline{\theta}_{i-1})$$

$$= \underline{k}_o (e^{j\Omega} - 1) + \underline{\theta}_i e^{j\Omega} - \underline{\theta}_{i-1}$$

$$= \underline{k}_o (e^{j\Omega} - 1) + (\alpha \underline{\theta}_{i-1} + \underline{q}_i) e^{j\Omega} - \underline{\theta}_{i-1}$$

$$= \underline{k}_o (e^{j\Omega} - 1) + \underline{\theta}_{i-1} [\alpha e^{j\Omega} - 1] + \underline{q}_i e^{j\Omega}.$$

~~(b)~~

$$\cancel{\tilde{\underline{k}}_i - \underline{c}_i e^{j\Omega(i-1)} = \tilde{\underline{k}}_{i-1} - \mu \underline{y}_i g(E_i)}$$

$$\cancel{\underline{y}_i^H \tilde{\underline{k}}_i - \underline{y}_i^H \underline{c}_i e^{j\Omega(i-1)} = \underline{y}_i^H \tilde{\underline{k}}_{i-1} - \mu \|\underline{y}_i\|^2 g(E_i)}$$

(b) $\tilde{\underline{k}}_i = \underline{k}_{o,i} e^{j\Omega i} - \underline{k}_{i-1} - \mu \underline{y}_i g(E_i)$

$$\underline{y}_i^H \tilde{\underline{k}}_i = \underline{y}_i^H \left(\underline{k}_{o,i} e^{j\Omega i} - \underline{k}_{i-1} \right) - \mu \|\underline{y}_i\|^2 g(E_i)$$

$$E_{pi}^* = E_{ai}^* - \mu \|\underline{y}_i\|^2 g(E_i)$$

From (a) $\rightarrow$ $\tilde{\underline{k}}_i = \tilde{\underline{k}}_{i-1} - \mu \underline{y}_i g(E_i) + \underline{c}_i e^{j\Omega(i-1)}$

(let $\underline{y}_i \neq \underline{0}$)

$$\tilde{\underline{k}}_i - \underline{c}_i e^{j\Omega(i-1)} = \tilde{\underline{k}}_{i-1} - \mu \underline{y}_i \left(\frac{E_{ai}^* - E_{pi}^*}{\mu \|\underline{y}_i\|^2} \right)$$

$$\tilde{\underline{k}}_i - \underline{c}_i e^{j\Omega(i-1)} = \tilde{\underline{k}}_{i-1} - \frac{\underline{y}_i}{\|\underline{y}_i\|^2} \left(E_{ai}^* - E_{pi}^* \right)$$

$$\tilde{\underline{k}}_i - \underline{c}_i e^{j\Omega(i-1)} - \frac{\underline{y}_i}{\|\underline{y}_i\|^2} E_{pi}^* = \tilde{\underline{k}}_{i-1} - \frac{\underline{y}_i}{\|\underline{y}_i\|^2} E_{ai}^*$$

$$\tilde{\underline{k}}_i - \underline{c}_i e^{j\Omega(i-1)} + \frac{\underline{y}_i}{\|\underline{y}_i\|^2} E_{ai}^* = \tilde{\underline{k}}_{i-1} + \frac{\underline{y}_i}{\|\underline{y}_i\|^2} E_{pi}^*$$

$$\Rightarrow \left\| \tilde{\underline{k}}_i - \underline{c}_i e^{j\Omega(i-1)} + \frac{\underline{y}_i}{\|\underline{y}_i\|^2} E_{ai}^* \right\|^2 = \left\| \tilde{\underline{k}}_{i-1} + \frac{\underline{y}_i}{\|\underline{y}_i\|^2} E_{pi}^* \right\|^2$$

$$\Rightarrow \|\tilde{\underline{k}}_i - \underline{c}_i e^{j\Omega(i-1)}\|^2 + \frac{|E_{ai}|^2}{\|\underline{y}_i\|^2} + \left(\tilde{\underline{k}}_i - \underline{c}_i e^{j\Omega(i-1)}\right)^H \frac{\underline{y}_i E_{ai}^*}{\|\underline{y}_i\|^2} + \frac{E_{ai} \underline{y}_i^H (\tilde{\underline{k}}_i - \underline{c}_i e^{j\Omega(i-1)})}{\|\underline{y}_i\|^2}$$

$$= \|\tilde{\underline{k}}_{i-1}\|^2 + \frac{|E_{pi}|^2}{\|\underline{y}_i\|^2} + \tilde{\underline{k}}_{i-1}^H \frac{\underline{y}_i E_{pi}^*}{\|\underline{y}_i\|^2} + \frac{E_{pi} \underline{y}_i^H \tilde{\underline{k}}_{i-1}}{\|\underline{y}_i\|^2}.$$

~~Using $\tilde{\underline{k}}_i^H \underline{y}_i = E_{ai}$ (Note ...)~~

Simplifying, we got

$$\|\tilde{\underline{k}}_i - \underline{c}_i e^{j\Omega(i-1)}\|^2 + \frac{|E_{ai}|^2}{\|\underline{y}_i\|^2} = \|\tilde{\underline{k}}_{i-1}\|^2 + \frac{|E_{pi}|^2}{\|\underline{y}_i\|^2} \quad \text{— (1)}$$

When $\underline{y}_i = \underline{0}$, $E_{a_i} = E_{p_i} = 0$. & $\tilde{\underline{K}}_i = \tilde{\underline{K}}_{i-1} + \underline{c}_i e^{j\Omega(i-1)}$

$$\Rightarrow \|\tilde{\underline{K}}_{i-1}\|^2 = \|\tilde{\underline{K}}_i - \underline{c}_i e^{j\Omega(i-1)}\|^2 \quad -② $$

Combining ① & ②, we get

$$\|\tilde{\underline{K}}_i - \underline{c}_i e^{j\Omega(i-1)}\|^2 + \bar{\mu}_i |E_{a_i}|^2 = \|\tilde{\underline{K}}_{i-1}\|^2 + \bar{\mu}_i |E_{p_i}|^2$$

where $\bar{\mu}_i = \begin{cases} \frac{1}{\|\underline{y}_i\|^2} & \underline{y}_i \neq \underline{0} \\ 0 & \underline{y}_i = \underline{0} \end{cases}$

(c) $\underline{K}_{o,i} = \underline{k}_o + \underline{\theta}_i$

$\underline{K}_{o,i-1} = \underline{k}_o + \underline{\theta}_{i-1}$

$\Rightarrow \underline{K}_{o,i} = \underline{K}_{o,i-1} + \underline{\theta}_i - \underline{\theta}_{i-1}$

$$E_i = x_i - \underline{K}_{i-1}^H \underline{y}_i = \underline{K}_{o,i}^H \underline{y}_i e^{-j\Omega i} + V_i - \underline{K}_{i-1}^H \underline{y}_i$$

$$= E_{a_i} + V_i$$

$$= \left(\underline{K}_{o,i} e^{+j\Omega i} - \underline{K}_{i-1}\right)^H \underline{y}_i + V_i$$

$$= \left(\tilde{\underline{K}}_{i-1} + \underline{K}_{o,i} e^{j\Omega i} - \underline{K}_{o,i-1} e^{j\Omega(i-1)}\right)^H \underline{y}_i + V_i$$

$$= \tilde{\underline{K}}_{i-1}^H \underline{y}_i + \left(\underline{K}_{o,i} e^{j\Omega} - \underline{K}_{o,i-1}\right)^H e^{-j\Omega(i-1)} \underline{y}_i + V_i$$

(d) We get the non-stationary model in class by setting $\Omega = 0$ and $\alpha = 1$.

② Energy conservation relation

$$\|\tilde{\underline{K}}_i - \underline{c}_i e^{j\Omega(i-1)}\|^2 + \bar{\mu}_i |E_{a_i}|^2 = \|\tilde{\underline{K}}_{i-1}\|^2 + \bar{\mu}_i |E_{p_i}|^2$$

Assume $\{\underline{q}_i\}$ i.i.d with covariance Q & indep. of $\underline{K}_{-1}, \underline{\theta}_{-1}, \{\underline{y}_j\}$ and x_j for $j < i$. $-$Ⓐ

Assume in steady state $E[\|\tilde{\underline{K}}_i\|^2] = E[\|\tilde{\underline{K}}_{i-1}\|^2]$ (as $i \to \infty$)

$$E\left[\|\tilde{\underline{k}}_i - \underline{c}_i e^{j\Omega(i-1)}\|^2\right] + E\left[\bar{\mu}_i |E_{ai}|^2\right] = E\left[\|\tilde{\underline{k}}_{i-1}\|^2\right] + E\left[\bar{\mu}_i |E_{pi}|^2\right]$$

$$E\left[\|\tilde{\underline{k}}_i - \underline{c}_i e^{j\Omega(i-1)}\|^2\right] = E\left[\|\tilde{\underline{k}}_i\|^2\right] + E\left[\|\underline{c}_i\|^2\right] - E\left[\tilde{\underline{k}}_i^H \underline{c}_i e^{j\Omega(i-1)}\right] - E\left[\underline{c}_i^H \tilde{\underline{k}}_i e^{-j\Omega(i-1)}\right]$$

$$= E\left[\|\tilde{\underline{k}}_i\|^2\right] + E\left[\|\underline{c}_i\|^2\right] - 2\,\mathrm{Re}\left[E\left[\underline{c}_i^H \tilde{\underline{k}}_i e^{-j\Omega(i-1)}\right]\right].$$

$$\Rightarrow E\left[\|\tilde{\underline{k}}_i\|^2\right] + E\left[\|\underline{c}_i\|^2\right] - 2\,\mathrm{Re}\left(E\left[\underline{c}_i^H \tilde{\underline{k}}_i e^{-j\Omega(i-1)}\right]\right) + E\left[\bar{\mu}_i |E_{ai}|^2\right] = E\left[\|\tilde{\underline{k}}_{i-1}\|^2\right] + E\left[\bar{\mu}_i |E_{pi}|^2\right] \quad \text{— (1)}$$

$$\tilde{\underline{k}}_i = \tilde{\underline{k}}_{i-1} - \mu \underline{y}_i g(E_i) + \underline{c}_i e^{j\Omega(i-1)}$$

$$\Rightarrow \underline{c}_i^H \tilde{\underline{k}}_i e^{-j\Omega(i-1)} = \underline{c}_i^H \tilde{\underline{k}}_{i-1} e^{-j\Omega(i-1)} - \mu \underline{c}_i^H \underline{y}_i g(E_i) e^{-j\Omega(i-1)} + \|\underline{c}_i\|^2.$$

$$\Rightarrow E\left[\underline{c}_i^H \tilde{\underline{k}}_i e^{-j\Omega(i-1)}\right] = E\left[\underline{c}_i^H \tilde{\underline{k}}_{i-1} e^{-j\Omega(i-1)}\right] - \mu E\left[\underline{c}_i^H \underline{y}_i g(E_i) e^{-j\Omega(i-1)}\right] + E\left[\|\underline{c}_i\|^2\right].$$

Substituting this in (1), we get

$$E\left[\|\tilde{\underline{k}}_i\|^2\right] - 2\,\mathrm{Re}\left(E\left[\underline{c}_i^H \tilde{\underline{k}}_{i-1} e^{-j\Omega(i-1)}\right]\right) - \mu\, 2\,\mathrm{Re}\left(E\left[\underline{c}_i^H \underline{y}_i g(E_i) e^{-j\Omega(i-1)}\right]\right) - E\left[\|\underline{c}_i\|^2\right] = E\left[\|\tilde{\underline{k}}_{i-1}\|^2\right] + E\left[\bar{\mu}_i |E_{pi}|^2\right] - E\left[\bar{\mu}_i |E_{ai}|^2\right]$$

In steady stat

$$-2\,\mathrm{Re}\left[E\left[\underline{c}_i^H \tilde{\underline{k}}_{i-1} e^{-j\Omega(i-1)}\right]\right] + 2\mu\,\mathrm{Re}\left[E\left[\underline{c}_i^H \underline{y}_i g(E_i) e^{-j\Omega(i-1)}\right]\right] - E\left[\|\underline{c}_i\|^2\right] = E\left[\bar{\mu}_i |E_{pi}|^2\right] - E\left[\bar{\mu}_i |E_{ai}|^2\right]. \quad \text{— (2)}$$

$$E_{pi} = E_{ai} - \mu g^*(E_i) \|\underline{y}_i\|^2$$

$$\Rightarrow \bar{\mu}_i |E_{pi}|^2 = \bar{\mu}_i \left|E_{ai} - \mu g^*(E_i) \|\underline{y}_i\|^2\right|^2$$

$$= \bar{\mu}_i |E_{ai}|^2 + \bar{\mu}_i \mu^2 |g(E_i)|^2 \|\underline{y}_i\|^4 - \bar{\mu}_i 2\,\mathrm{Re}\left(\mu \|\underline{y}_i\|^2 g^*(E_i) E_{ai}^*\right)$$

$$= \overline{\mu}_i |E_{a_i}|^2 + \mu^2 |g(E_i)|^2 \|\underline{y}_i\|^2 - 2\mu \,\mathrm{Re}\left(g^*(E_i) E_{a_i}^*\right)$$

Substituting in (2), we get :

$$-2\,\mathrm{Re}\left[E\left[\underline{c}_i^H \tilde{\underline{k}}_{i-1} e^{-j\Omega(i-1)}\right]\right] + 2\mu\,\mathrm{Re}\left[E\left[\underline{c}_i^H \underline{y}_i g(E_i) e^{-j\Omega(i-1)}\right]\right] - E\left[\|\underline{c}_i\|^2\right]$$

$$= +\mu^2 E\left[|g(E_i)|^2 \|\underline{y}_i\|^2\right] - 2\mu\,\mathrm{Re}\left(E\left[g^*(E_i) E_{a_i}^*\right]\right)$$

$$\Rightarrow 2\,\mathrm{Re}\left(E\left[g^*(E_i)E_{a_i}^*\right]\right) = \mu E\left[|g(E_i)|^2 \|\underline{y}_i\|^2\right]$$
$$+ 2\,\mathrm{Re}\left(E\left[\underline{c}_i^H \tilde{\underline{k}}_{i-1} e^{-j\Omega(i-1)}\right]\right)\mu^{-1}$$
$$- 2\mu\,\mathrm{Re}\left[E\left[\underline{c}_i^H \underline{y}_i g(E_i) e^{-j\Omega(i-1)}\right]\right)\mu^{-1}$$
$$+ E\left[\|\underline{c}_i\|^2\right]\mu^{-1}$$

$$\Rightarrow 2\,\mathrm{Re}\left(E\left[g^*(E_i)E_{a_i}^*\right]\right) = \mu E\left[|g(E_i)|^2 \|\underline{y}_i\|^2\right]$$
$$+ 2\mu^{-1}\mathrm{Re}\left(E\left[\underline{c}_i^H \left(\tilde{\underline{k}}_{i-1} - \mu \underline{y}_i g(E_i)\right)\right] e^{-j\Omega(i-1)}\right)$$
$$+ E\left[\|\underline{c}_i\|^2\right]\mu^{-1} \quad \text{— (3)}.$$

$$\underline{c}_i = \underline{k}_0 (e^{j\Omega} - 1) + \underline{\theta}_{i-1} (\alpha e^{j\Omega} - 1) + \underline{q}_i e^{j\Omega}$$

$$\Rightarrow \underline{c}_i^H\left(\tilde{\underline{k}}_{i-1} - \mu \underline{y}_i g(E_i)\right) = \underline{k}_0^H\left(\tilde{\underline{k}}_{i-1} - \mu \underline{y}_i g(E_i)\right)\left(e^{-j\Omega} - 1\right)$$
$$+ \underline{\theta}_{i-1}^H\left(\tilde{\underline{k}}_{i-1} - \mu \underline{y}_i g(E_i)\right)\left(\alpha^* e^{-j\Omega} - 1\right)$$
$$+ \underline{q}_i^H\left(\tilde{\underline{k}}_{i-1} - \mu \underline{y}_i g(E_i)\right) e^{-j\Omega}$$

Substituting in (3), we get

$$2\,\mathrm{Re}\left(E\left[g^*(E_i) E_{a_i}^*\right]\right) = \mu E\left[|g(E_i)|^2 \|\underline{y}_i\|^2\right] - 2\mu^{-1}\mathrm{Re}\left[(1 - e^{-j\Omega}) E\left[\underline{k}_0^H\left(\tilde{\underline{k}}_{i-1} - \mu \underline{y}_i g(E_i)\right)\right] e^{-j\Omega(i-1)}\right]$$
$$- 2\mu^{-1}\mathrm{Re}\left[(1 - \alpha^* e^{-j\Omega}) E\left[\underline{\theta}_{i-1}^H\left(\tilde{\underline{k}}_{i-1} - \mu \underline{y}_i g(E_i)\right)\right] e^{-j\Omega(i-1)}\right]$$
$$+ 2\mu^{-1}\mathrm{Re}\left[E\left[\underline{q}_i^H\left(\tilde{\underline{k}}_{i-1} - \mu \underline{y}_i g(E_i)\right)\right] \underbrace{e^{-j\Omega} e^{-j\Omega(i-1)}}_{e^{-j\Omega i}}\right]$$
$$+ \mu^{-1} E\left[\|\underline{c}_i\|^2\right] \quad \text{— (4)}$$

Using the assumptions on q_i we can show

$$E\left[q_i^H\left(\tilde{k}_{i-1}-\mu \underline{y}_i g(E_i)\right)\right]=0$$

$$\& \quad E\left[\|\underline{s}_i\|^2\right]=\|\underline{k}_0\|^2\left|1-e^{j\Omega}\right|^2+E\left[\|\underline{\theta}_{i-1}\|^2\right]\left|1-\alpha e^{j\Omega}\right|^2+E\left[\|q_i\|^2\right]$$

$$=\|\underline{k}_0\|^2\left|1-e^{j\Omega}\right|^2+\mathrm{Tr}(\Theta)\left|1-\alpha e^{j\Omega}\right|^2+\mathrm{Tr}(Q)$$

(where $Q=E[q_i q_i^H]$, $\Theta=E[\underline{\theta}_i\underline{\theta}_i^H]$ as $i\to\infty$).

$\Rightarrow$ (4) simplifies to

$$\boxed{\begin{aligned}2\,\mathrm{Re}\left(E\left[g^*(E_i)E_{a_i}^*\right]\right) &= \mu E\left[|g(E_i)|^2\|\underline{y}_i\|^2\right]+\mu^{-1}\mathrm{Tr}(Q)\\ &+\mu^{-1}\mathrm{Tr}(\Theta)\left|1-\alpha e^{j\Omega}\right|^2+\mu^{-1}\|\underline{k}_0\|^2\left|1-e^{j\Omega}\right|^2\\ &-2\mu^{-1}\mathrm{Re}\left((1-e^{-j\Omega})E\left[\underline{k}_0^H(\tilde{k}_{i-1}-\mu\underline{y}_i g(E_i))\right]e^{-j\Omega(i-1)}\right)\\ &-2\mu^{-1}\mathrm{Re}\left((1-\alpha^* e^{-j\Omega})E\left[\underline{\theta}_{i-1}^H(\tilde{k}_{i-1}-\mu\underline{y}_i g(E_i))\right]e^{-j\Omega(i-1)}\right)\end{aligned}}$$

For $\alpha=1$, $\Omega=0$, this simplifies to

$$\boxed{2\,\mathrm{Re}\left(E\left[g^*(E_i)E_{a_i}^*\right]\right)=\mu E\left[|g(E_i)|^2\|\underline{y}_i\|^2\right]+\mu^{-1}\mathrm{Tr}(Q)}$$ (derived in class).