EE5040 Problem Set 4

(1) $V_i = U_i + \alpha U_{i-1} + \beta U_{i-2}$ for $i > -\infty$ (Second order AR)

roots of $1 + \alpha z^{-1} + \beta z^{-2} = 0$ are strictly inside the unit circle

& α, β are real.

$$\underline{y} = \begin{bmatrix} U_{i-1} \\ U_{i-2} \end{bmatrix}, \qquad \hat{U}_i = \underline{k}^H \underline{y} \qquad V_i \longrightarrow \boxed{\frac{1}{1+\alpha z^{-1}+\beta z^{-2}}} \longrightarrow U_i$$

$$U_i = V_i - \alpha U_{i-1} - \beta U_{i-2}.$$

(a) Let z_1, z_2 be the roots of $z^2 + \alpha z + \beta = 0$.

z_1, z_2 are strictly inside the unit circle.

$$\Rightarrow |z_1 z_2| = |\beta| < 1 \Rightarrow \underline{|\beta| < 1}.$$

$$\left.\begin{array}{l} z_1 + z_2 = \alpha \text{ (real)} \\ z_1 z_2 = \beta \text{ (real)} \end{array}\right\} \Rightarrow z_1 = z_2^*.$$

$$\Rightarrow |z_1|^2 = \beta \quad \text{(or)} \quad |z_1| = \sqrt{\beta}.$$

$$\Rightarrow |z_1 + z_2| < 2\sqrt{\beta} \leq 1 + \beta$$

$$\text{i.e. } \underline{|\alpha| \leq 1 + \beta}.$$

(b) $$R_y = E\left(\begin{bmatrix} U_{i-1} \\ U_{i-2} \end{bmatrix} \begin{bmatrix} U_{i-1}^* & U_{i-2}^* \end{bmatrix}\right) = \begin{bmatrix} R_U(0) & R_U(1) \\ R_U^*(1) & R_U(0) \end{bmatrix}$$

$$R_{yU_i} = E\left[\begin{pmatrix} U_{i-1} \\ U_{i-2} \end{pmatrix} U_i^*\right] = \begin{bmatrix} R_U^*(1) \\ R_U^*(2) \end{bmatrix}.$$

$$R_{U_i y} = R_{yU_i}^H.$$

Need to find $R_U(0)$, $R_U(1)$, $R_U(2)$.

$\{U_i\}$ is an AR(2) process.

The autocorrelation of an AR(2) process is given as follows

(Ref: Modern Spectral Estimation: Theory & Application, S. M. Kay, Prentice Hall, 1988)
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$$H(z) = \frac{1}{1+\alpha z^{-1}+\beta z^{-2}} \qquad \{v_i\} \longrightarrow \boxed{H(z)} \longrightarrow \{u_i\}$$

Let $z_1 = re^{j\theta}$. Then, $z_2 = re^{-j\theta}$

$$z_1 + z_2 = \alpha \Rightarrow 2r\cos\theta = \alpha$$

$$z_1 z_2 = \beta \Rightarrow r^2 = \beta$$

$$R_u(k) = \sigma_v^2 \underbrace{\left[\frac{\frac{1+\beta}{1-\beta}\sqrt{1+\left(\frac{1-\beta}{1+\beta}\right)^2 \cot^2\theta}}{1-2\beta\cos 2\theta+\beta^2}\right]}_{\triangleq A} r^{|k|}\cos(\theta|k|-\psi)$$

where $\psi = \tan^{-1}\left[\frac{1-\beta}{1+\beta}\cot\theta\right]$.

$$R_u(0) = \sigma_v^2 A \cos(-\psi) = \sigma_v^2 A\cos\psi$$

$$R_u(1) = \sigma_v^2 A \; r\cos(\theta-\psi)$$

$$R_u(2) = \sigma_v^2 A \; r^2\cos(2\theta-\psi)$$

$$R_y = \begin{bmatrix} R_u(0) & R_u(1) \\ R_u^*(1) & R_u(0) \end{bmatrix} = A\sigma_v^2 \begin{bmatrix} \cos\psi & r\cos(\theta-\psi) \\ r\cos(\theta-\psi) & \cos\psi \end{bmatrix}$$

$$R_{yu_i} = \begin{bmatrix} r\cos(\theta-\psi) \\ r^2\cos(2\theta-\psi) \end{bmatrix} A\sigma_v^2$$

Note that since $|\beta|<1$ & $|\alpha|<1+\beta$, $(1-\beta)\left[(1+\beta)^2-\alpha^2\right]>0$.

(c) $u_i = v_i - \alpha u_{i-1} - \beta u_{i-2}$

We observe u_{i-1} & u_{i-2} and predict u_i. (all zero-mean)

$$\hat{u}_i = \hat{v}_i - \alpha\hat{u}_{i-1} - \beta\hat{u}_{i-2} \quad \text{(estimates given } u_{i-1} \text{ \& } u_{i-2}\text{)}$$

$$= 0 - \alpha u_{i-1} - \beta u_{i-2} \Rightarrow k_{opt} = \begin{bmatrix} -\alpha \\ -\beta \end{bmatrix}.$$

(d) $R_y = A\sigma_v^2 \begin{bmatrix} \cos\psi & r\cos(\theta-\psi) \\ r\cos(\theta-\psi) & \cos\psi \end{bmatrix}$

Suppose $H = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ $(a>0)$, then its eigenvalues satisfy

$$(a-\lambda)^2 - b^2 = 0.$$

$$\Rightarrow a = \lambda \pm b \quad (\text{or}) \ \lambda = a \pm b$$

$$\Rightarrow \text{Eigen value spread} = \frac{a+|b|}{a-|b|}.$$

Here
$a = \cos\psi, \quad b = r\cos(\theta-\psi) = r\cos\theta\cos\psi + r\sin\theta\sin\psi$

$$= [r\cos\theta + r\sin\theta\tan\psi]\cos\psi.$$

$$\text{Eigenvalue spread of } R_y = \frac{a+|b|}{a-|b|}$$

$$= \frac{\cos\psi\,[1 + |r\cos\theta + r\sin\theta\tan\psi|]}{\cos\psi\,[1 - |r\cos\theta + r\sin\theta\tan\psi|]}$$

$$\left[\tan\psi = \frac{1-\beta}{1+\beta}\cot\theta \Rightarrow r\sin\theta\tan\psi = \left(\frac{1-\beta}{1+\beta}\cos\theta\right)r\right]$$

$$= \frac{1 + |r\cos\theta|\,|1 + \frac{1-\beta}{1+\beta}|}{1 - |r\cos\theta|\,|1 + \frac{1-\beta}{1+\beta}|}$$

$$= \frac{1 + \frac{|\alpha|}{2}\frac{2}{1+\beta}}{1 - \frac{|\alpha|}{2}\frac{2}{1+\beta}} = \frac{(1+\beta)+|\alpha|}{(1+\beta)-|\alpha|}.$$

Steepest descent algorithm

$$\underline{k}_{i+1} = \underline{k}_i + \mu\,[R_{yv_i} - R_y\underline{k}_i].$$

where $0 < \mu < \frac{2}{\lambda_{max}}$ for convergence.

$$\lambda_{max} = A\sigma_v^2(a+|b|)$$
$$\lambda_{min} = A\sigma_v^2(a-|b|).$$

$$A = \frac{\frac{1+\beta}{1-\beta}\sqrt{1+\left(\frac{1-\beta}{1+\beta}\right)^2\cot^2\theta}}{1-2\beta\cos 2\theta+\beta^2}$$

Need $\mu < \dfrac{2}{\lambda_{max}} = \dfrac{2}{A\sigma_v^2(a+|b|)}$

$a = \cos\psi$
$b = r\cos(\theta-\psi)$
$\psi = \tan^{-1}\left[\frac{1-\beta}{1+\beta}\cot\theta\right]$

$$a+|b| = \cos\psi\left[1+\frac{|\alpha|}{1+\beta}\right].$$

$$\frac{2}{A\sigma_v^2(a+|b|)} = \frac{2}{\sigma_v^2(Aa+A|b|)} = \frac{2}{\sigma_v^2(Aa)\left(1+\frac{|b|}{a}\right)}$$

$$= \frac{2}{\sigma_v^2\left(\frac{1+\beta}{1-\beta}\right)[(1+\beta)^2-\alpha^2]\left[1+\frac{|\alpha|}{1+\beta}\right]}$$

$$= \frac{2}{\sigma_v^2\left(\frac{1}{1-\beta}\right)\left((1+\beta)^2-\alpha^2\right)\left(1+\beta+|\alpha|\right)}$$

$$\boxed{\mu < \frac{2(1-\beta)}{\sigma_v^2\left((1+\beta)^2-\alpha^2\right)\left(1+\beta+|\alpha|\right)}}.$$

(e) $$\mu^\circ = \frac{2}{\lambda_{max}+\lambda_{min}} = \frac{2}{A\sigma_v^2(2a)} = \frac{1}{A\sigma_v^2 a} = \frac{1}{\sigma_v^2}\left[\frac{1}{Aa}\right].$$

$$A = \frac{\frac{1+\beta}{1-\beta}\sqrt{1+\left(\frac{1-\beta}{1+\beta}\right)^2\cot^2\theta}}{1-2\beta\cos 2\theta+\beta^2}$$

$$a = \cos\varphi = \cos\left(\tan^{-1}\left(\frac{(1-\beta)\cos\theta}{(1+\beta)\sin\theta}\right)\right) = \sqrt{\frac{(1+\beta)^2\sin^2\theta}{(1-\beta)^2\cos^2\theta+(1+\beta)^2\sin^2\theta}} = \frac{1}{\sqrt{1+\left(\frac{1-\beta}{1+\beta}\right)^2\cot^2\theta}}$$

$$\Rightarrow Aa = \frac{\frac{1+\beta}{1-\beta}}{1-2\beta\cos 2\theta+\beta^2}$$

Note that $\alpha = 2\sqrt{\beta}\cos\theta \Rightarrow \alpha^2 = 4\beta\cos^2\theta$

$$\Rightarrow (1+\beta)^2-\alpha^2 = 1+\beta^2+2\beta-\alpha^2$$
$$= 1+\beta^2+2\beta(1-2\cos^2\theta)$$
$$= 1+\beta^2-2\beta\cos 2\theta$$

$$\Rightarrow \frac{1}{Aa} = \frac{1-\beta}{1+\beta}(1-2\beta\cos 2\theta+\beta^2) = \frac{1-\beta}{1+\beta}\left((1+\beta)^2-\alpha^2\right)$$

$$\Rightarrow \boxed{\mu^\circ = \frac{(1+\beta)^2-\alpha^2}{\sigma_v^2}\cdot\frac{1-\beta}{1+\beta}}.$$

$$\Rightarrow \tau^\circ = \frac{-1}{2\ln|1-\mu^\circ\lambda_{max}|}$$

$$\mu^\circ\lambda_{max} = \frac{2\lambda_{max}}{\lambda_{max}+\lambda_{min}} \Rightarrow 1-\mu^\circ\lambda_{max} = \frac{\lambda_{min}-\lambda_{max}}{\lambda_{max}+\lambda_{min}}$$

$$\Rightarrow |1-\mu^\circ\lambda_{max}| = \frac{\lambda_{max}-\lambda_{min}}{\lambda_{max}+\lambda_{min}} = \frac{a}{|b|}$$

$$= \frac{1}{\left(\frac{|\alpha|}{1+\beta}\right)} = \frac{1+\beta}{|\alpha|}.$$

$$\Rightarrow \tau^{\circ} = \frac{-1}{2\ln\left(\frac{1+\beta}{|\alpha|}\right)} = \frac{1}{2\ln\left(\frac{|\alpha|}{1+\beta}\right)}.$$

(2) $\underline{k}_{i+1} = \underline{k}_i + \mu_i \left[R_{yx} - R_y \underline{k}_i \right]$

$\mu_i \longrightarrow \alpha > 0$ as $i \to \infty$ $\qquad \tilde{\underline{k}}_{i+1} = \underline{k}_{opt} - \underline{k}_{i+1}$

and $\alpha < \dfrac{2}{\lambda_{max}}$.

$$\tilde{\underline{k}}_{i+1} = \tilde{\underline{k}}_i - \mu_i \left[R_y \underline{k}_{opt} - R_y \underline{k}_i \right]$$

$$\tilde{\underline{k}}_{i+1} = \left(I - \mu_i R_y \right) \tilde{\underline{k}}_i$$

$$\underline{x}_{i+1} = \left(I - \mu_i \Lambda \right) \underline{x}_i \qquad \left(\text{where } U^H \tilde{\underline{k}}_i \triangleq \underline{x}_i \right)$$

$$x_{i+1,k} = \left(1 - \mu_i \lambda_k\right) x_{i,k} \quad \text{for each } k$$

$$= \left[\prod_{l=0}^{i} (1 - \mu_l \lambda_k) \right] x_{0,k}$$

$\prod_{l=0}^{i} (1 - \mu_l \lambda_k)$ should $\to 0$ as $i \to \infty$ for each k.

$\mu_i \to \alpha > 0$ implies that $\alpha - \epsilon < \mu_i < \alpha + \epsilon$ for any $\epsilon > 0$ for sufficiently large i.

Also, ϵ can be chosen such that $0 < \alpha - \epsilon$ and $\alpha + \epsilon < \dfrac{2}{\lambda_{max}}$.

$\Rightarrow$ We have $\mu_i > \alpha - \epsilon > 0$ for $i > i_0$ sufficiently large.

$$\Rightarrow \mu_i \lambda_k > (\alpha - \epsilon)\lambda_k > 0 \quad \text{for } i > i_0$$

$$\Rightarrow 1 - \mu_i \lambda_k < 1 - (\alpha - \epsilon)\lambda_k \quad \text{for } i > i_0.$$

Since $\alpha + \epsilon < \dfrac{2}{\lambda_{max}}$, we also have $\mu_i < \dfrac{2}{\lambda_{max}} \Rightarrow 1 - \mu_i \lambda_k > -1$

$$\Rightarrow |1 - \mu_i \lambda_k| < \underbrace{|1 - (\alpha - \epsilon)\lambda_k|}_{\text{strictly less than } 1} \quad \text{for } i > i_0.$$

$$\Rightarrow \prod_{l=0}^{i} (1 - \mu_l \lambda_k) \to 0 \quad \text{as } i \to \infty \text{ for each } k.$$

(3) $$\mu_i^o = \frac{\underline{\tilde{k}}_i^H R_y^2 \underline{\tilde{k}}_i}{\underline{\tilde{k}}_i^H R_y^3 \underline{\tilde{k}}_i} \qquad R_y > 0.$$

$$\cancel{\lambda_{min}^2 \|\underline{\tilde{k}}_i\|^2 \le \underline{\tilde{k}}_i^H R_y^2 \underline{\tilde{k}}_i \le \lambda_{max}^2 \|\underline{\tilde{k}}_i\|^2} \quad \text{Let } R_y \underline{\tilde{k}}_i = \underline{x}_i$$

$$\cancel{\& \ \lambda_{min}^3 \|\underline{\tilde{k}}_i\|^2 \le \underline{\tilde{k}}_i^H R_y^3 \underline{\tilde{k}}_i \le \lambda_{max}^3 \|\underline{\tilde{k}}_i\|^2}$$

$$\underline{\tilde{k}}_i^H R_y^2 \underline{\tilde{k}}_i = \left(\underline{\tilde{k}}_i^H R_y\right)\left(R_y \underline{\tilde{k}}_i\right) = \|R_y \underline{\tilde{k}}_i\|^2 = \underline{x}_i^H \underline{x}_i$$

$$\underline{\tilde{k}}_i^H R_y^3 \underline{\tilde{k}}_i = \left(\underline{\tilde{k}}_i^H R_y\right) R_y \left(R_y \underline{\tilde{k}}_i\right) = \underline{x}_i^H R_y \underline{x}_i$$

$$\Rightarrow \mu_i^o = \frac{\underline{x}_i^H \underline{x}_i}{\underline{x}_i^H R_y \underline{x}_i}$$

We know that $\lambda_{min} \le \dfrac{\underline{x}_i^H R_y \underline{x}_i}{\underline{x}_i^H \underline{x}_i} \le \lambda_{max}$

$\Rightarrow \mu_i^o$ satisfies $\underline{\dfrac{1}{\lambda_{max}} \le \mu_i^o \le \dfrac{1}{\lambda_{min}}}$

Since $\dfrac{1}{\lambda_{max}} > 0$ & $\mu_i^o > \dfrac{1}{\lambda_{max}}$ for all i $\underline{\sum_i \mu_i^o \text{ diverges.}}$

(4) Note that $\mu_i^o = \dfrac{\underline{\tilde{k}}_i^H R_y^2 \underline{\tilde{k}}_i}{\underline{\tilde{k}}_i^H R_y^3 \underline{\tilde{k}}_i}$ & $-\nabla_{\underline{k}_i} J(\underline{k}_i)^H = R_y \underline{\tilde{k}}_i$

$$\Rightarrow \mu_i^o = \frac{\nabla_{\underline{k}_i} J(\underline{k}_i) \nabla_{\underline{k}_i} J(\underline{k}_i)^H}{\nabla_{\underline{k}_i} J(\underline{k}_i) R_y \nabla_{\underline{k}_i} J(\underline{k}_i)^H}$$