EE5040 Problem Set 3

(1) $X = X_c + Z$ $\quad X, \underline{Y}$ zero-mean

$\hat{X} = K_0 \underline{Y}$ where $K_0 R_Y = R_{XY}$

$$R_{XY} = E\left[X \underline{Y}^H\right] = E\left[(X_c + Z)\underline{Y}^H\right] = E\left[X_c \underline{Y}^H\right] + E\left[Z\underline{Y}^H\right]$$

$$= E\left[X_c \underline{Y}^H\right] = R_{X_cY}.$$

Therefore, $K_0 R_Y = R_{X_cY}$.

$$\hat{X}_c = K_0 \underline{Y} = \hat{X}.$$

[Even if $\underline{Y}$ was not zero mean, we will get the same result.

$$\hat{X} = K_0(\underline{Y} - E[\underline{Y}])$$

where $K_0 R_Y = R_{XY}$ $\quad R_Y = E\left[(\underline{Y} - E[\underline{Y}])(\underline{Y} - E[\underline{Y}])^H\right]$

$$R_{XY} = E\left[X_c(\underline{Y} - E[\underline{Y}])^H\right] = R_{X_cY}.$$

$$\Rightarrow \hat{X}_c = \hat{X}\ ]$$

(2) $\underline{\hat{X}} = K_0 \underline{Y}$ where K_0 is any solution to $K_0 R_Y = R_{XY}$.

We want to minimize $E\left[\underline{\tilde{X}}'^H W \underline{\tilde{X}}'\right]$ for some $W \geq 0$. $(\underline{\tilde{X}}' = \underline{X} - \underline{\hat{X}}')$

$\underline{\hat{X}}' = K\underline{Y}$.

$$E\left[\underline{\tilde{X}}'^H W \underline{\tilde{X}}'\right] = E\left[(\underline{X} - K\underline{Y})^H W (\underline{X} - K\underline{Y})\right]$$

$$= E\left[(\underline{X} - K_0\underline{Y} + K_0\underline{Y} - K\underline{Y})^H W (\underline{X} - K_0\underline{Y} + K_0\underline{Y} - K\underline{Y})\right]$$

$$= E\left[(\underline{X} - K_0\underline{Y})^H W (\underline{X} - K_0\underline{Y})\right] + E\left[(\underline{X} - K_0\underline{Y})^H W (K_0\underline{Y} - K\underline{Y})\right] \quad (1)$$

$$+ E\left[(K_0\underline{Y} - K\underline{Y})^H W (\underline{X} - K_0\underline{Y})\right] + E\left[(K_0\underline{Y} - K\underline{Y})^H W (K_0\underline{Y} - K\underline{Y})\right]$$

We know that $K_0\underline{Y}$ is the linear MMSE estimate of $\underline{X}$ from $\underline{Y}$.

Therefore, we know $\underbrace{(\underline{X} - K_0\underline{Y})}_{\text{error}} \perp \underline{Y}$, ie., $E\left[(\underline{X} - K_0\underline{Y})\underline{Y}^H\right] = 0$

Let $\underline{\tilde{X}} = \underline{X} - K_0\underline{Y}$. $\quad E\left[\underline{\tilde{X}}\,\underline{Y}^H\right] = 0 \quad \Rightarrow E\left[\tilde{X}_i Y_j^*\right] = 0 \quad \forall\, i, j.$