EE5040 Problem Set 1

① $\underline{y}_i = X + V_i \quad i = 0, 1, \ldots, N-1$

(a) V_i, IID, $\sim N(0, \sigma_v^2)$

X ± 1 with equal probability.

$$P[X = \pm 1 / \underline{Y} = \underline{y}] = \frac{f_{\underline{Y}}(\underline{y}/X=+1)\, P[X=+1]}{f_{\underline{Y}}(\underline{y})}$$

$$= \frac{f_{\underline{Y}}(\underline{y}|X=+1)\, P[X=+1]}{f_{\underline{Y}}(\underline{y}|X=+1)\, P[X=+1] + f_{\underline{Y}}(\underline{y}/X=-1)\, P(X=-1)}.$$

Here $P[X=+1] = P[X=-1] = \frac{1}{2}$.

$f_{\underline{Y}}(\underline{y}/X=+1]$ is $N(\underline{1}, \sigma_v^2 I)$

$f_{\underline{Y}}(\underline{y}|X=-1]$ is $N(-\underline{1}, \sigma_v^2 I)$

$$f_{\underline{Y}}(\underline{y}/X=+1) = \frac{1}{(2\pi)^{N/2} |\sigma_v^2 I|^{1/2}} e^{-\frac{1}{2}\left[(\underline{y}-\underline{1})^T (\sigma_v^2 I)^{-1} (\underline{y}-\underline{1})\right]}$$

$$f_{\underline{Y}}(\underline{y}|X=-1) = \frac{1}{(2\pi)^{N/2} |\sigma_v^2 I|^{1/2}} e^{-\frac{1}{2}\left[(\underline{y}+\underline{1})^T (\sigma_v^2 I)^{-1} (\underline{y}+\underline{1})\right]}$$

$$P[X=+1/\underline{Y}=\underline{y}] = \frac{f_{\underline{Y}}(\underline{y}|X=+1)}{f_{\underline{Y}}(\underline{y}|X=+1) + f_{\underline{Y}}(\underline{y}/X=-1)}$$

$$= \frac{e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(y_n-1)^2}}{e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(y_n-1)^2} + e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(y_n+1)^2}}$$

$$E[X/\underline{Y}=\underline{y}] = P[X=+1/\underline{Y}=\underline{y}] - \underbrace{P[X=-1/\underline{Y}=\underline{y}]}_{=1-P[X=+1/\underline{Y}=\underline{y}]}$$

$$= \frac{e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(y_n-1)^2} - e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(y_n+1)^2}}{e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(y_n-1)^2} + e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(y_n+1)^2}}$$

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$$= \frac{e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(-2y_n)} - e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(2y_n)}}{e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(-2y_n)} + e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(2y_n)}}$$

$$= \tanh\left(\frac{\sum_{n=0}^{N-1} y_n}{\sigma_v^2}\right)$$

Therefore, $\boxed{\hat{X}_N = \tanh\left(\frac{\sum_{n=0}^{N-1} y_n}{\sigma_v^2}\right)}$

(b) $$P[X=+1/\underline{Y}=\underline{y}] = \frac{f_{\underline{Y}}(\underline{y}/X=+1)\,P[X=+1]}{f_{\underline{Y}}(\underline{y}/X=+1)\,P(X=+1)+f_{\underline{Y}}(\underline{y}/X=-1)\,P(X=-1)}$$

$$= \frac{p\,f_{\underline{Y}}(\underline{y}/X=+1)}{p f_{\underline{Y}}(\underline{y}/X=+1)+(1-p)f_{\underline{Y}}(\underline{y}/X=-1)}$$

$$E[X/\underline{Y}=\underline{y}] = P[X=+1/\underline{Y}=\underline{y}] - \left(1-P[X=+1/\underline{Y}=\underline{y}]\right)$$

$$= \frac{p f_{\underline{Y}}(\underline{y}/X=+1) - (1-p) f_{\underline{Y}}(\underline{y}/X=-1)}{p f_{\underline{Y}}(\underline{y}/X=+1) + (1-p) f_{\underline{Y}}(\underline{y}/X=-1)} \quad \text{— } 1$$

$$= \frac{p\, e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(y_n-1)^2} - (1-p)\, e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(y_n+1)^2}}{p\, e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(y_n-1)^2} + (1-p)\, e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(y_n+1)^2}}$$

$$= \frac{p\, e^{+\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n} - (1-p)\, e^{-\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n}}{p\, e^{+\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n} + (1-p)\, e^{-\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n}}$$

$$= \frac{e^{\ln p}\, e^{\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n} - e^{\ln(1-p)}\, e^{-\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n}}{e^{\ln p}\, e^{\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n} + e^{\ln(1-p)}\, e^{-\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n}}$$

$$= \frac{e^{\frac{+\ln p + \ln(1-p)}{2}}\left[e^{\frac{\ln p - \ln(1-p)}{2}} e^{\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n} - e^{\frac{-\ln p + \ln(1-p)}{2}} e^{\frac{-1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n}\right]}{e^{\frac{\ln p + \ln(1-p)}{2}}\left[e^{\frac{\ln p - \ln(1-p)}{2}} e^{\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n} - e^{\frac{-\ln p + \ln(1-p)}{2}} e^{-\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n}\right]}$$

$$= \frac{e^{\left[\frac{1}{2}\ln\frac{p}{1-p} + \frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n\right]} - e^{\left[-\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n - \frac{1}{2}\ln\frac{p}{1-p}\right]}}{e^{\left[\frac{1}{2}\ln\frac{p}{1-p} + \frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n\right]} + e^{\left[-\frac{1}{2}\ln\frac{p}{1-p} - \frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n\right]}}$$

$$= \tanh\left(\frac{1}{2}\ln\frac{p}{1-p} + \frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n\right)$$

$$\Rightarrow \boxed{\hat{X}_N = \tanh\left(\frac{1}{2}\ln\frac{p}{1-p} + \sum_{n=0}^{N-1}\frac{y_n}{\sigma_v^2}\right)}$$

(c) Starting from 1 in previous page, we have

$$E[X|\underline{Y}=\underline{y}] = \frac{p f_{\underline{Y}}(\underline{y}|X=+1) - (1-p) f_{\underline{Y}}(\underline{y}|X=-1)}{p f_{\underline{Y}}(\underline{y}|X=+1) + (1-p) f_{\underline{Y}}(\underline{y}|X=-1)}$$

$$f_{\underline{Y}}(\underline{y}|X=+1) = \frac{1}{(2\pi)^{N/2}|R_v|^{1/2}} e^{-\frac{1}{2}\left((\underline{y}-\underline{1})^T R_v^{-1}(\underline{y}-\underline{1})\right)}$$

$$f_{\underline{Y}}(\underline{y}|X=-1) = \frac{1}{(2\pi)^{N/2}(R_v)^{1/2}} e^{-\frac{1}{2}\left[(\underline{y}+\underline{1})^T R_v^{-1}(\underline{y}+\underline{1})\right]}$$

$$(\underline{y}-\underline{1})^T R_v^{-1}(\underline{y}-\underline{1}) = \underline{y}^T R_v^{-1}\underline{y} \overbrace{- \underline{y}^T R_v^{-1}\underline{1} - \underline{1}^T R_v^{-1}\underline{y}}^{=-2\,\underline{1}^T R_v^{-1}\underline{y}} + \underline{1}^T R_v^{-1}\underline{1}$$

$$(\underline{y}+\underline{1})^T R_v^{-1}(\underline{y}+\underline{1}) = \underline{y}^T R_v^{-1}\underline{y} + \underbrace{\underline{y}^T R_v^{-1}\underline{1} + \underline{1}^T R_v^{-1}\underline{y}}_{=2\,\underline{1}^T R_v^{-1}\underline{y}} + \underline{1}^T R_v^{-1}\underline{1}$$

$$E[X|\underline{Y}=\underline{y}] = \frac{p\, e^{+\underline{1}^T R_v^{-1}\underline{y}} - (1-p)\, e^{-\underline{1}^T R_v^{-1}\underline{y}}}{p\, e^{+\underline{1}^T R_v^{-1}\underline{y}} + (1-p)\, e^{-\underline{1}^T R_v^{-1}\underline{y}}}$$

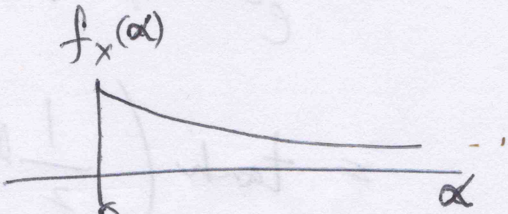
$$\begin{pmatrix}\text{Simplification}\\ \text{as in part(b)}\end{pmatrix} = \tanh\left(\frac{1}{2}\ln\frac{p}{1-p} + \underline{1}^T R_V^{-1}\underline{y}\right).$$

$$\boxed{\hat{X}_N = \tanh\left(\frac{1}{2}\ln\frac{p}{1-p} + \underline{1}^T R_V^{-1}\underline{y}\right)}$$

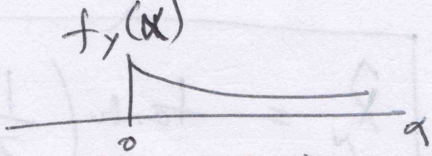
(2) $X \sim f_X(x) = \lambda_1 e^{-\lambda_1 x}$ for $x \geq 0$

$V \sim f_V(v) = \lambda_2 e^{-\lambda_2 v}$ for $v \geq 0$

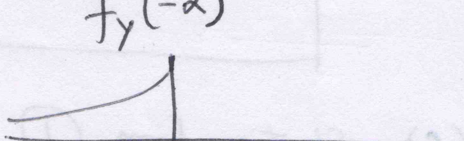
X and V are independent.



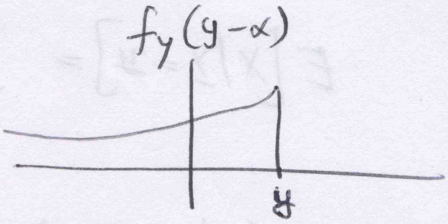
(a) $Y = X + V$



$$f_Y(y) = \int_0^{y} f_X(\alpha)\, f_Y(y-\alpha)\, d\alpha$$



$$= \int_0^{y} \lambda_1 e^{-\lambda_1\alpha} \cdot \lambda_2 e^{-\lambda_2(y-\alpha)}\, d\alpha$$



$$= \lambda_1\lambda_2 \int_0^{y} e^{-(\lambda_1-\lambda_2)\alpha}\, e^{-\lambda_2 y}\, d\alpha$$

$f_X(\alpha) f_Y(y-\alpha)$ is non-zero only for $0 \leq \alpha \leq y$.

$$= \lambda_1\lambda_2 e^{-\lambda_2 y} \int_0^{y} e^{-(\lambda_1-\lambda_2)\alpha}\, d\alpha$$

$$= \lambda_1\lambda_2 e^{-\lambda_2 y} \left.\frac{e^{-(\lambda_1-\lambda_2)\alpha}}{-(\lambda_1-\lambda_2)}\right|_0^{y}$$

$$= \lambda_1\lambda_2 e^{-\lambda_2 y}\left[\frac{e^{-(\lambda_1-\lambda_2)y}}{-(\lambda_1-\lambda_2)} - \frac{1}{-(\lambda_1-\lambda_2)}\right]$$

$$= \frac{\lambda_1\lambda_2 e^{-\lambda_2 y}}{\lambda_2-\lambda_1}\left[e^{-(\lambda_1-\lambda_2)y} - 1\right] \quad \underline{\text{for } y \geq 0.}$$

b) $Y = X + V$ X and V are independent

$$f_{Y|X=x}(y) = f_{V|X=x}(y-x) = f_V(y-x)$$

$$= \lambda_2 e^{-\lambda_2(y-x)}$$

$$f_{X,Y}(x,y) = f_X(x)\, f_{Y|X=x}(y)$$

$$= \lambda_1 e^{-\lambda_1 x} \cdot \lambda_2 e^{-\lambda_2(y-x)}$$

$$= \lambda_1 \lambda_2 e^{-(\lambda_1-\lambda_2)x} \cdot e^{-\lambda_2 y} \quad \text{for } x \geq 0 \;\&\; y \geq 0 \;\&\; x \leq y.$$

c) $\hat{X} = E[X \mid Y=y]$

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{\lambda_1 \lambda_2 e^{(\lambda_2-\lambda_1)x - \lambda_2 y}}{\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1} e^{-\lambda_2 y}\left[e^{(\lambda_2-\lambda_1)y} - 1\right]} \quad \text{for } x \geq 0 \;\&\; y \geq x.$$

$$= \frac{(\lambda_2-\lambda_1)\, e^{(\lambda_2-\lambda_1)x}}{\left[e^{(\lambda_2-\lambda_1)y} - 1\right]}.$$

$$\hat{X} = \int_0^y x \cdot f_{X|Y=y}(x)\, dx$$

$$= \frac{\lambda_2-\lambda_1}{\left[e^{(\lambda_2-\lambda_1)y} - 1\right]} \int_0^y x\, e^{(\lambda_2-\lambda_1)x}\, dx$$

$$\int_0^y x e^{(\lambda_2-\lambda_1)x} dx = \int_0^y x \, d\left(\frac{e^{(\lambda_2-\lambda_1)x}}{(\lambda_2-\lambda_1)}\right)$$

$$= \left.\frac{x e^{(\lambda_2-\lambda_1)x}}{(\lambda_2-\lambda_1)}\right|_0^y - \int_0^y \frac{e^{(\lambda_2-\lambda_1)x}}{\lambda_2-\lambda_1} dx$$

$$= \frac{y e^{(\lambda_2-\lambda_1)y}}{\lambda_2-\lambda_1} - \left.\frac{e^{(\lambda_2-\lambda_1)x}}{(\lambda_2-\lambda_1)^2}\right|_0^y$$

$$= \frac{y e^{(\lambda_2-\lambda_1)y}}{(\lambda_2-\lambda_1)} - \frac{e^{(\lambda_2-\lambda_1)y}}{(\lambda_2-\lambda_1)^2} + \frac{1}{(\lambda_2-\lambda_1)^2}$$

$$\hat{X} = \frac{(\lambda_2-\lambda_1)}{\left[e^{(\lambda_2-\lambda_1)y}-1\right]}\left[\frac{y e^{(\lambda_2-\lambda_1)y}}{\lambda_2-\lambda_1} - \frac{e^{(\lambda_2-\lambda_1)y}}{(\lambda_2-\lambda_1)^2} + \frac{1}{(\lambda_2-\lambda_1)^2}\right]$$

$$= \frac{1}{e^{(\lambda_2-\lambda_1)y}-1}\left[y e^{(\lambda_2-\lambda_1)y} - \frac{e^{(\lambda_2-\lambda_1)y}}{\lambda_2-\lambda_1} + \frac{1}{\lambda_2-\lambda_1}\right]$$

$$= \frac{1}{e^{(\lambda_2-\lambda_1)y}-1}\left[y\left(e^{(\lambda_2-\lambda_1)y}-1\right) + y - \frac{\left(e^{(\lambda_2-\lambda_1)y}-1\right)}{\lambda_2-\lambda_1}\right]$$

$$= \left[y + \frac{y}{e^{(\lambda_2-\lambda_1)y}-1} - \frac{1}{\lambda_2-\lambda_1}\right]$$

$$= \frac{1}{\lambda_1-\lambda_2} + \left[1 + \frac{1}{e^{(\lambda_2-\lambda_1)y}-1}\right]y$$

$$= \frac{1}{\lambda_1-\lambda_2} + \frac{e^{(\lambda_2-\lambda_1)y}}{e^{(\lambda_2-\lambda_1)y}-1}\,y.$$

$$= \frac{1}{\lambda_1-\lambda_2} + \frac{e^{-\lambda_1 y}}{e^{-\lambda_1 y}-e^{-\lambda_2 y}}\,y$$

$$= \frac{1}{\lambda_1-\lambda_2} - \frac{e^{-\lambda_1 y}}{e^{-\lambda_2 y}-e^{-\lambda_1 y}}\,y.$$

———— X ————