EE5040: Adaptive Signal Processing
Problem Set 7: Recursive least-squares

1. (Sayed VII.12, QR method) Consider the normal equations $H^H H\hat{\mathbf{w}} = H^H\mathbf{x}$, where H is $N\times M$. Assume H has full column rank (i.e., the rank of H is M) so that $H^H H$ is positive definite. A method to reduce the effects of ill-conditioning of H on the solution of the normal equations is to avoid forming the product $H^H H$ and to determine the Cholesky factor L by working directly with H. This can be achieved by appealing to the so-called QR decomposition of H namely,
$$H = Q\begin{bmatrix} R \\ 0 \end{bmatrix},$$
where Q is $N\times N$ unitary and R is $M\times M$ upper-triangular with positive diagonal entries.
   (a) Show that $L = R^H$. (Remark: With the L so determined, we can solve the normal equations by using the two-step procedure in problem set 6. Alternatively, we can proceed as below, which is nowadays the preferred way of solving the normal equations due to its numerical reliability.)
   (b) Let $\begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} = Q^H\mathbf{x}$, where $\mathbf{z}_1$ is $M\times 1$. Verify that $||\mathbf{x} - H\mathbf{w}||^2 = ||\mathbf{z}_1 - R\mathbf{w}||^2 + ||\mathbf{z}_2||^2$.Conclude that the solution $\hat{\mathbf{w}}$ can be obtained by solving the triangular linear system of equations $R\hat{\mathbf{w}} = \mathbf{z}_1$. Conclude further that the resulting minimum cost is $||\mathbf{z}_2||^2$.

2. (Sayed, Chapter 34, Lemma 34.2, Complex Givens rotation) Consider a 1×2 vector $[a\ b]$ with possibly complex entries. Then choose Θ as
$$\Theta = \frac{1}{\sqrt{1+|\rho|^2}}\begin{bmatrix} 1 & -\rho \\ \rho^* & 1 \end{bmatrix} \quad \text{where} \quad \rho = \frac{b}{a},\ a \neq 0$$
to get
$$[a\ b]\Theta = \pm e^{j\phi_a}\sqrt{|a|^2+|b|^2}[1\ 0]$$
where ϕ_a denotes the phase of a. If $a = 0$, then choose Θ as
$$\Theta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
to get $[0\ b]\Theta = [b\ 0]$.

3. (Sayed, Example 34.1, Using Givens rotations) Assume we are given a 2×3 matrix A
$$A = \begin{bmatrix} 1 & 0.75 & 0.75 \\ 0.4 & 0.2 & 0.2 \end{bmatrix}$$
and that we wish to reduce it to the form
$$A\Theta = \begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \end{bmatrix}$$
via a sequence of Givens rotations. (Hint: First make the (1, 3) entry 0, then the (1, 2) entry, and finally the (2, 3) entry.)