EE5040: Adaptive Signal Processing
Problem Set 1: Optimal least-mean-squares estimation

1. (Sayed I.13) Consider noisy observations $Y_i = X + V_i$, where X and V_i are independent real-valued random variables, V_i is a white-noise Gaussian random process with zero mean and variance σ_v^2, and X takes the values ± 1 with equal probability. The value of X is the same for all measurements $\{Y_i\}$.

   (a) Show that the least-mean-squares estimate of X in terms of $\{Y_0, Y_1, \cdots, Y_{N-1}\}$ is

   $$\hat{X}_N = \tanh\left(\sum_{i=0}^{N-1} \frac{Y_i}{\sigma_v^2}\right).$$

   (b) Assume X takes the value 1 with probability p and the value -1 with probability $1-p$. Show that the least-mean-squares estimate of X in terms of $\{Y_0, Y_1, \cdots, Y_{N-1}\}$ is

   $$\hat{X}_N = \tanh\left(\frac{1}{2}\ln\left(\frac{p}{1-p}\right) + \sum_{i=0}^{N-1} \frac{Y_i}{\sigma_v^2}\right).$$

   (c) Assume that the noise is correlated. Let $\mathbf{R}_v = E[\mathbf{V}\mathbf{V}^T]$, where $\mathbf{V} = [V_0 \;\; V_1 \;\; \cdots \;\; V_{N-1}]^T$. Show that the least-mean-squares estimate of X in terms of $\{Y_0, Y_1, \cdots, Y_{N-1}\}$ is

   $$\hat{X}_N = \tanh\left(\frac{1}{2}\ln\left(\frac{p}{1-p}\right) + \mathbf{1}^T\mathbf{R}_v^{-1}\mathbf{Y}\right),$$

   where $\mathbf{1} = [1 \;\; 1 \;\; \cdots \;\; 1]^T_{N\times 1}$.

2. (Sayed I.16) Suppose we observe $Y = X + V$, where X and V are independent real-valued random variables with exponential distributions with parameters λ_1 and λ_2 ($\lambda_1 \neq \lambda_2$). That is, the PDFs of X and V are $f_X(x) = \lambda_1 e^{-\lambda_1 x}$ for $x \geq 0$ and $f_V(v) = \lambda_2 e^{-\lambda_2 v}$ for $v \geq 0$, respectively.

   (a) Using the fact that the PDF of the sum of two independent random variables is the convolution of the individual PDFs, show that

   $$f_Y(y) = \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_2 y}\left[e^{(\lambda_2-\lambda_1)y} - 1\right], \quad y \geq 0.$$

   (b) Establish that $f_{X,Y}(x, y) = \lambda_1\lambda_2 e^{(\lambda_2-\lambda_1)x-\lambda_2 y}$, for $x \geq 0$ and $y \geq 0$.

   (c) Show that the least-mean-squares estimate of X given $Y = y$ is

   $$\hat{X} = \frac{1}{\lambda_1 - \lambda_2} - \frac{e^{-\lambda_1 y}}{e^{-\lambda_2 y} - e^{-\lambda_1 y}} y.$$