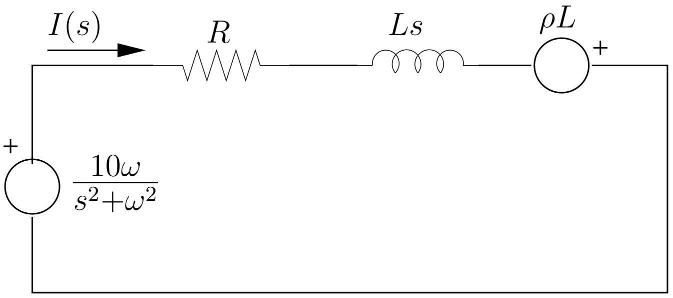
EC204: Networks & Systems
Solution to Problem Set 7

1. The transformed network is shown below.



$I(s)$ can be determined as follows.

$$I(s) = \frac{\frac{10\omega}{s^2+\omega^2} + \rho L}{R + Ls} = \frac{\rho}{s + \frac{R}{L}} + \frac{10\omega/L}{(s^2+\omega^2)\left(s+\frac{R}{L}\right)}$$

The second term of $I(s)$ can be expanded as

$$\frac{10\omega/L}{(s^2+\omega^2)\left(s+\frac{R}{L}\right)} = \frac{K_1}{s+\frac{R}{L}} + \frac{K_2 s + K_3}{s^2+\omega^2}.$$

$$K_1 = \frac{10\omega L}{R^2 + \omega^2 L^2}$$

K_2 and K_3 can be found by equating the coefficients of s^2 and s in the numerator of the left hand side and right hand side of the expansion. Therefore, we get

$$K_1 + K_2 = 0 \quad \text{and} \quad K_2 \frac{R}{L} + K_3 = 0,$$

leading to

$$K_2 = -K_1 \quad \text{and} \quad K_3 = -K_2 \frac{R}{L}.$$

Therefore, we have

$$I(s) = \frac{\rho}{s+\frac{R}{L}} + \frac{10\omega L}{R^2+\omega^2L^2}\frac{1}{s+\frac{R}{L}} + \left(-\frac{10\omega L}{R^2+\omega^2L^2}\right)\frac{1}{s^2+\omega^2} + \frac{10\omega R}{R^2+\omega^2L^2}\frac{1}{s^2+\omega^2},$$

and

$$i(t) = \rho e^{-Rt/L} + \frac{10}{R^2+\omega^2L^2}\left[\omega L e^{-Rt/L} - \omega L\cos\omega t + R\sin\omega t\right]$$

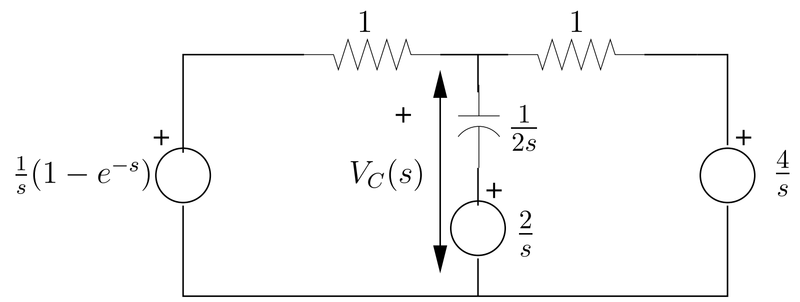
for $t \geq 0$. The total solution $i(t)$ can be split into its transient and steady state components as $i(t) = i_{tr}(t) + i_{ss}(t)$ where

$$i_{tr}(t) = \rho e^{-Rt/L} + \frac{10\omega L}{R^2 + \omega^2 L^2} e^{-Rt/L}$$

and

$$i_{ss}(t) = \frac{10}{R^2 + \omega^2 L^2} \left[-\omega L \cos \omega t + R \sin \omega t \right].$$

2. The condition at $t = 0^-$ can be easily obtained as $v_C(0^-) = 2V$. Then, the transformed network is as shown below.



From the above network, we have

$$V_C(s) - \frac{4}{s} + \left(V_C(s) - \frac{2}{s} \right) 2s + V_C(s) - \frac{1}{s}(1 - e^{-s}) = 0.$$

Therefore, we have

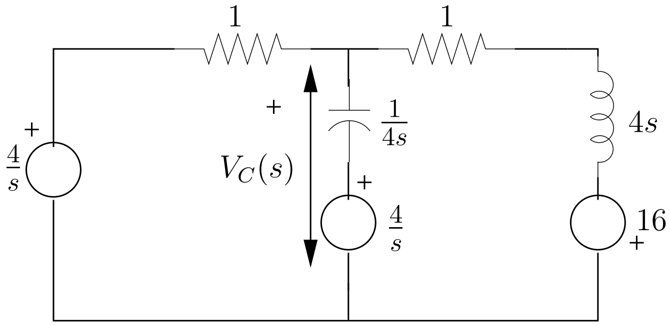
$$\begin{aligned} V_C(s) &= \frac{\frac{4}{s} + 4 + \frac{1}{s}(1 - e^{-s})}{1 + 2s + 1} \\ &= \frac{4s + 4 + (1 - e^{-s})}{s(2s + 2)} \\ &= \frac{2}{s} + \frac{1}{2s(s+1)}(1 - e^{-s}) \\ &= \frac{2}{s} + \left[\frac{1/2}{s} + \frac{-1/2}{s+1} \right] (1 - e^{-s}). \end{aligned}$$

Finally, we have

$$v_C(t) = 2u(t) + \frac{1}{2}(1 - e^{-t})u(t) - \frac{1}{2}(1 - e^{-(t-1)})u(t-1)$$

for $t \geq 0$.

3. The conditions at $t = 0^-$ can be easily obtained as $v_C(0^-) = 4V$ and $i_L(0^-) = 4A$. Then, the transformed network is as shown below.



From the above network, we have

$$V_C(s) - \frac{4}{s} + \frac{V_C(s) + 16}{1 + 4s} + \left(V_C(s) - \frac{4}{s}\right) 4s = 0.$$

Therefore, we have

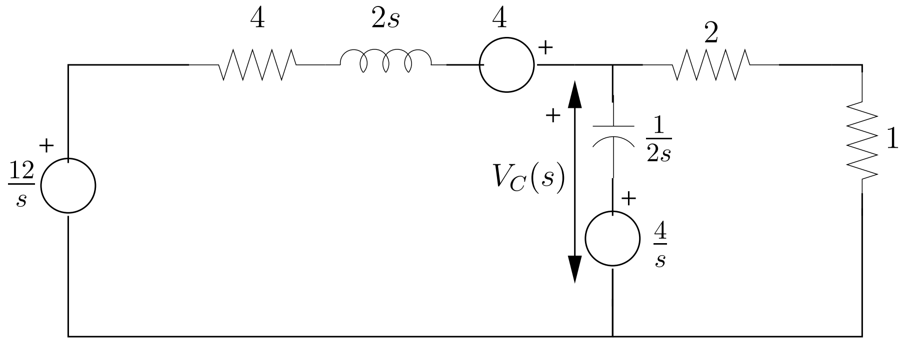
$$\begin{aligned} V_C(s) &= \frac{\frac{4}{s} + 16 + \frac{16}{1+4s}}{1 + \frac{1}{1+4s} + 4s} \\ &= \frac{2(16s^2 + 4s + 1)}{s(8s^2 + 4s + 1)} \\ &= \frac{2}{s}\left[1 + \frac{8s^2}{8s^2 + 4s + 1}\right] \\ &= \frac{2}{s} + \frac{2s}{\left(s + \frac{1}{4}\right)^2 + \frac{1}{16}}. \end{aligned}$$

Finally, we have

$$v_C(t) = 2 + 2e^{-t/4} \cos \frac{t}{4}$$

for $t \geq 0$.

4. The conditions at $t = 0^-$ can be easily obtained as $v_C(0^-) = 4V$ and $i_L(0^-) = 2A$. Then, the transformed network is as shown below.



From the above network, we have

$$\frac{V_C(s)}{3} + \left(V_C(s) - \frac{4}{s}\right) 2s + \frac{\left(V_C(s) - \frac{12}{s} - 4\right)}{4 + 2s} = 0.$$

Therefore, we have

$$V_C(s) = \frac{8 + \frac{\frac{12}{s}+4}{4+2s}}{\frac{1}{3} + 2s + \frac{1}{4+2s}} = \frac{12(4s^2 + 9s + 3)}{s(12s^2 + 26s + 7)}.$$

$$v_C(0^+) = \lim_{s\to\infty} sV_C(s) = 4.$$

For $t \geq 0^+$, we have

$$\frac{dv_C}{dt} \Longleftrightarrow sV_C(s) - v_C(0^+) = \frac{4s + 8}{12s^2 + 26s + 7}.$$

Therefore, we have (using initial value theorem)

$$\left.\frac{dv_C}{dt}\right|_{t=0^+} = \lim_{s\to\infty} s\left[\frac{4s + 8}{12s^2 + 26s + 7}\right] = \frac{1}{3}.$$

Similarly, we have

$$\frac{d^2v_C}{dt^2} \Longleftrightarrow s\mathcal{L}\left[\frac{dv_C}{dt}\right] - \left.\frac{dv_C}{dt}\right|_{t=0^+} = \frac{-2s - 7}{3(12s^2 + 26s + 7)}.$$

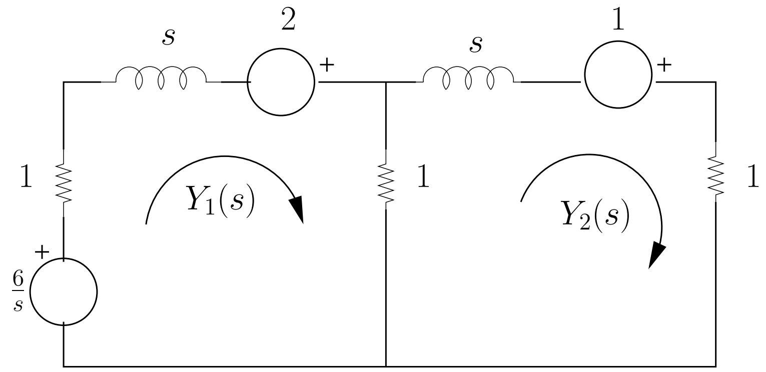
$$\left.\frac{d^2v_C}{dt^2}\right|_{t=0^+} = \lim_{s\to\infty} s\left[\frac{-2s - 7}{3(12s^2 + 26s + 7)}\right] = \frac{-1}{18}.$$

and

$$\frac{d^3v_C}{dt^3} \Longleftrightarrow s\mathcal{L}\left[\frac{d^2v_C}{dt^2}\right] - \left.\frac{d^2v_C}{dt^2}\right|_{t=0^+} = \frac{-204s - 21}{54(12s^2 + 26s + 7)}.$$

$$\left.\frac{d^3v_C}{dt^3}\right|_{t=0^+} = \lim_{s\to\infty} s\left[\frac{-204s - 21}{54(12s^2 + 26s + 7)}\right] = \frac{-17}{54}.$$

5. The initial conditions at $t = 0^-$ are $y_1(0^-) = 2A$ and $y_2(0^-) = 1A$. The transformed network is as shown below.



The loop equations are:

$$Y_1(s)\,[2+s] - Y_2(s) = 2 + \frac{6}{s}$$

and

$$Y_2(s)\,[2+s] - Y_1(s) = 1.$$

Solving these loop equations, we have

$$Y_1(s) = \frac{2s^2 + 11s + 12}{s(s+1)(s+3)},$$

and

$$Y_2(s) = \frac{s^2 + 4s + 6}{s(s+1)(s+3)}.$$

Using partial fraction expansion, we get

$$Y_1(s) = \frac{4}{s} + \frac{-1.5}{s+1} + \frac{-0.5}{s+3}$$

and

$$Y_2(s) = \frac{2}{s} + \frac{-1.5}{s+1} + \frac{0.5}{s+3}.$$

Therefore, we get

$$y_1(t) = 4u(t) - 1.5e^{-t}u(t) - 0.5e^{-3t}u(t),$$

and

$$y_2(t) = 2u(t) - 1.5e^{-t}u(t) + 0.5e^{-3t}u(t).$$

6. (a) $H(s) = \dfrac{s+3}{(s+2)^3} = \dfrac{s+2+1}{(s+2)^3} = \dfrac{1}{(s+2)^2} + \dfrac{1}{(s+2)^3}$. Therefore, the impulse response $h(t)$ is given by

$$h(t) = te^{-2t}u(t) + \frac{t^2}{2}e^{-2t}u(t).$$

(b) Steady state response to $10u(t)$ is $[H(s)|_{s=0}]\,10u(t)$.

$$H(s)|_{s=0} = \frac{3}{8}.$$

Therefore, the steady state response to $10u(t)$ is $3.75u(t)$.

(c) Steady state response to $e^{j2t}u(t)$ is $\left[H(s)|_{s=j2}\right] e^{j2t}u(t)$.

$$H(s)|_{s=j2} = \frac{-1-j5}{32} = -0.03125 - j0.15625.$$

Therefore, the steady state response to $e^{j2t}u(t)$ is $(-0.03125 - j0.15625)e^{j2t}u(t)$.

7. An input $x(t) = u(t)$ gives an output $y(t) = (4e^{-t} - 3e^{-2t})u(t)$.

$$X(s) = \frac{1}{s} \quad \text{and} \quad Y(s) = \frac{4}{s+1} - \frac{3}{s+2} = \frac{s+5}{(s+1)(s+2)}.$$

(b) System function $H(s) = \dfrac{Y(s)}{X(s)} = \dfrac{s(s+5)}{(s+1)(s+2)}$.

(a) Impulse response $h(t) = \mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}\left[1 - \dfrac{4}{s+1} + \dfrac{6}{s+2}\right] = \delta(t) - 4e^{-t}u(t) + 6e^{-2t}u(t)$.

(c) Let the output for the input $x(t) = e^{-4t}u(t)$ be $y(t)$.

$$Y(s) = X(s)H(s) = \frac{s(s+5)}{(s+4)(s+1)(s+2)} = \frac{-4/3}{s+1} + \frac{3}{s+2} + \frac{-2/3}{s+4}.$$

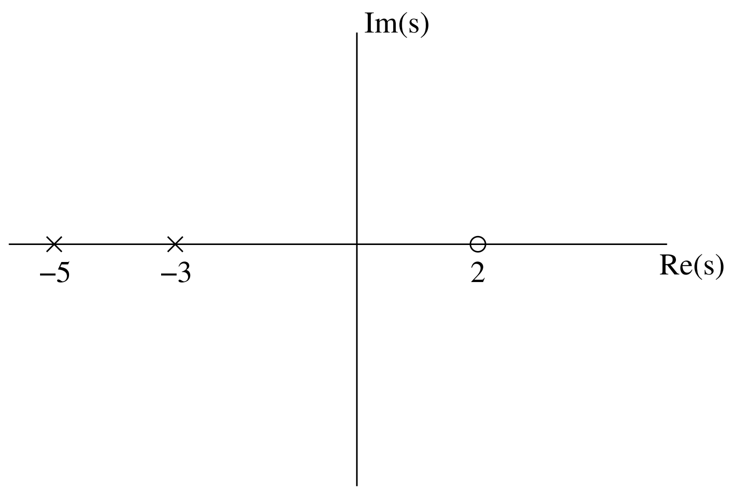
Therefore, $y(t) = \left[-\dfrac{4}{3}e^{-t} + 3e^{-2t} - \dfrac{2}{3}e^{-4t}\right] u(t)$.

(d) The steady state response to $\cos 2t$ is $[H(s)|_{s=j2}] \cos 2t$.

$$H(s)|_{s=j2} = \frac{j2(5+j2)}{(1+j2)(2+j2)} = 1.7e^{j\theta} \quad \text{where} \quad \theta = 3.4^o.$$

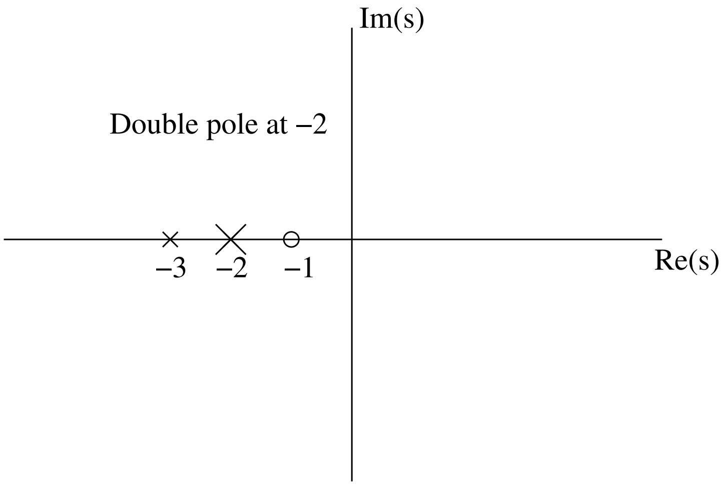
Therefore, the steady state response is $1.7\cos(2t + 3.4^o)$.

8. (a) Pole-Zero plot:



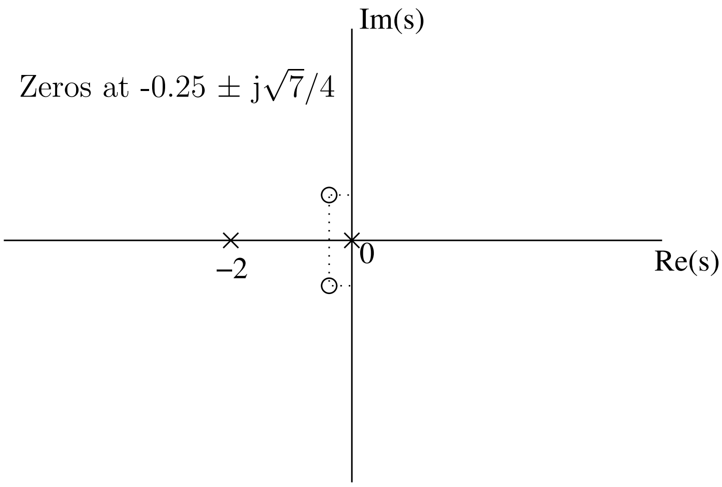
This system is BIBO stable.

(b) Pole-Zero plot:



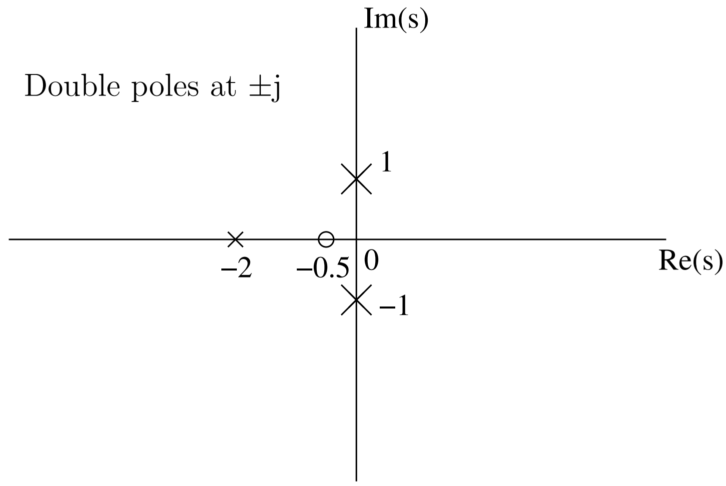
This system is BIBO stable.

(c) Pole-Zero plot:



This system is not BIBO stable.

(d) Pole-Zero plot:



This system is not BIBO stable.

9. $H(s) = \frac{K(s+a)}{(s+1-j)(s+1+j)} = \frac{K(s+a)}{(s+1)^2+1}$.
The Laplace transform of the output to a unit step input is

$$Y(s) = H(s)\frac{1}{s} = \frac{K(s+a)}{s((s+1)^2+1)} = K\left[\frac{A}{s} + \frac{Bs+C}{(s+1)^2+1}\right],$$

where

$$A = \left.\frac{s+a}{(s+1)^2+1}\right|_{s=0} = \frac{a}{2}.$$

Equating the coefficients of s in the numerator of $Y(s)$ and its partial fraction expansion, we have $A+B=0$ and $2A+C=1$. Therefore, we have

$$B = -\frac{a}{2},$$

and

$$C = 1-a.$$

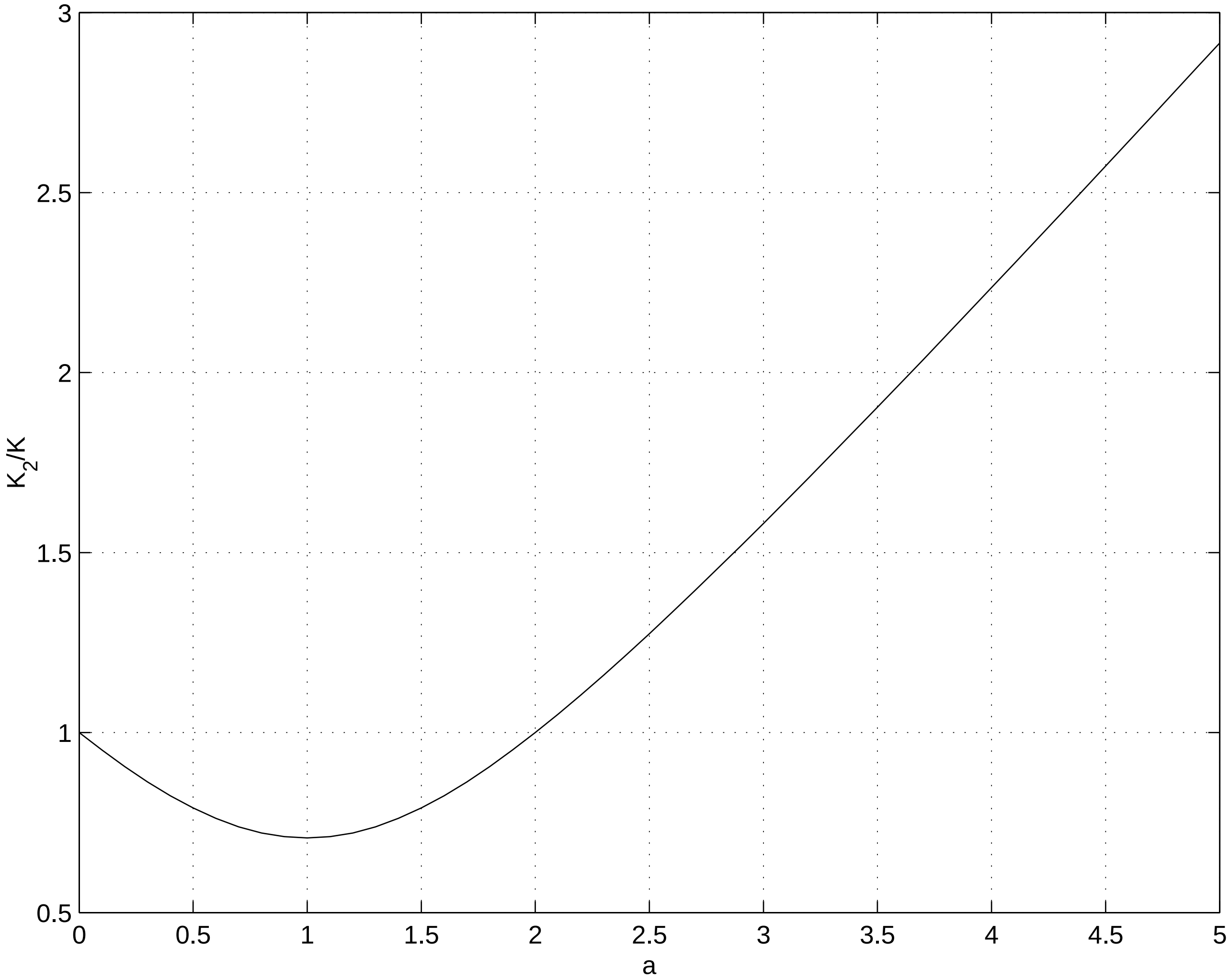
We want to determine the term in $y(t)$ of the form $K_2 e^{-t}\sin(t+\phi)$. The will correspond to the inverse laplace transform of the second term in the partial fraction expansion.

$$\mathcal{L}^{-1}\left[\frac{Bs+C}{(s+1)^2+1}\right] = \mathcal{L}^{-1}\left[\frac{B(s+1)+(C-B)}{(s+1)^2+1}\right]$$

$$= Be^{-t}\cos t + (C-B)e^{-t}\sin t = \sqrt{B^2+(C-B)^2}\sin(t+\phi).$$

Therfore, we have

$$K_2 = K\sqrt{B^2+(C-B)^2} = K\sqrt{\left(\frac{a}{2}\right)^2 + \left(1-\frac{a}{2}\right)^2}.$$



K_2/K
a
0.5
1
1.5
2
2.5
3
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5