EC204: Networks & Systems
Solutions to Problem Set 4

1. (a) $x(t)$ is real $\Rightarrow x(t) = x^*(t) \Rightarrow X(\omega) = X^*(-\omega) \Rightarrow X(-\omega) = X^*(\omega)$

$$x_e(t) = \frac{x(t) + x^*(-t)}{2} = \frac{x(t) + x(-t)}{2}$$
$$x_o(t) = \frac{x(t) - x^*(-t)}{2} = \frac{x(t) + x(-t)}{2}$$
$$x_e(t) \Longleftrightarrow \frac{X(\omega) + X(-\omega)}{2} = \frac{X(\omega) + X^*(\omega)}{2} = Re[X(\omega)]$$
$$x_o(t) \Longleftrightarrow \frac{X(\omega) - X(-\omega)}{2} = \frac{X(\omega) - X^*(\omega)}{2} = \jmath Im[X(\omega)]$$

(b) $x(t) = e^{-at}u(t) \Longleftrightarrow \dfrac{1}{a + \jmath\omega} = \dfrac{a}{a^2 + \omega^2} - \jmath \dfrac{\omega}{a^2 + \omega^2} = X(\omega)$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}e^{-at}u(t) + \frac{1}{2}e^{at}u(-t)$$

$$\therefore x_e(t) \Longleftrightarrow \frac{1}{2}\left[\frac{1}{a + \jmath\omega} + \frac{1}{a - \jmath\omega}\right] = \frac{a}{a^2 + \omega^2} = Re[X(\omega)]$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}e^{-at}u(t) - \frac{1}{2}e^{at}u(-t)$$

$$\therefore x_o(t) \Longleftrightarrow \frac{1}{2}\left[\frac{1}{a + \jmath\omega} - \frac{1}{a - \jmath\omega}\right] = \frac{-\jmath\omega}{a^2 + \omega^2} = \jmath Im[X(\omega)]$$

2. (a)

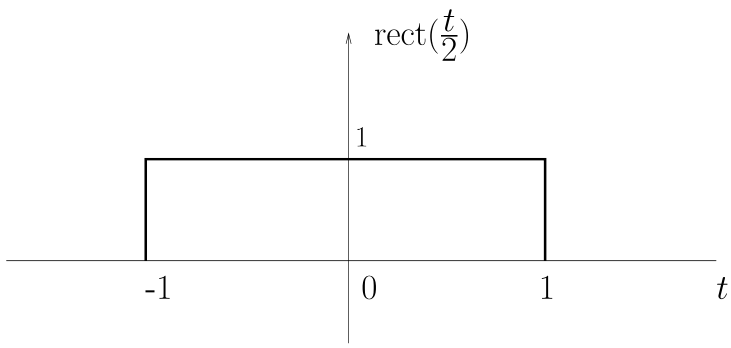


Figure 1: Problem 2(a)

(b)

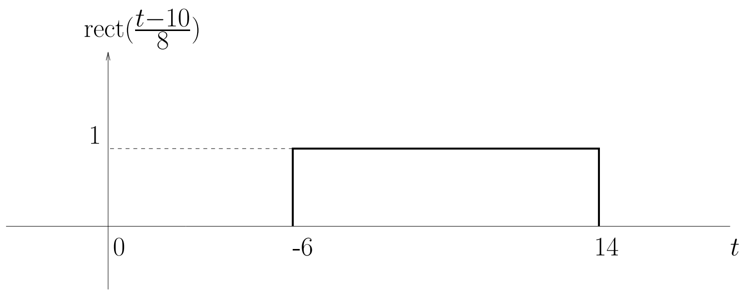


Figure 2: Problem 2(b)

(c)

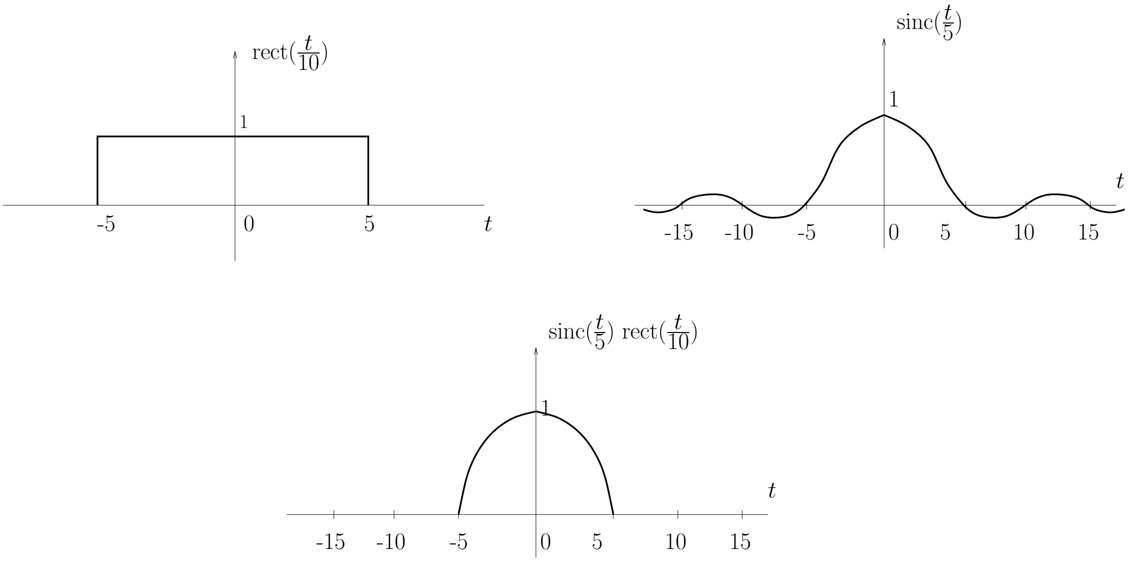


Figure 3: Problem 2(c)

3. $x(t) \Longleftrightarrow X(\omega)$, therefore we have

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-\jmath\omega t} d\omega$$

$$\Longrightarrow X(0) = \int_{-\infty}^{\infty} x(t) d\omega$$

Similarly, we have

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{\jmath\omega t} d\omega$$

$$\Longrightarrow x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) d\omega$$

We know that $rect(t) \Longleftrightarrow sinc\left(\frac{\omega}{2\pi}\right)$

$$\int_{-\infty}^{\infty} sinc(x) dx = ?$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} sinc\left(\frac{\omega}{2\pi}\right) d\omega = 1 \Longrightarrow \int_{-\infty}^{\infty} sinc(x) dx = 1$$

Similarly, $rect(t) * rect(t) \Longleftrightarrow sinc^2\left(\frac{\omega}{2\pi}\right)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} sinc^2\left(\frac{\omega}{2\pi}\right)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} sinc^2\left(x\right)dx(2\pi)$$

$$= \int_{-\infty}^{\infty} sinc^2\left(x\right)dx = rect(t) * rect(t)|_{t=0} = 1$$

$$\therefore \int_{-\infty}^{\infty} sinc^2\left(x\right)dx = 1$$

4.

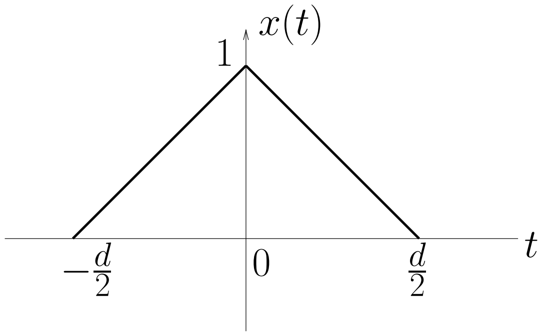


Figure 4: Problem 4

Method 1:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-\jmath\omega t} dt$$

$$= \int_{-d/2}^{0} \left(\frac{2t}{d} + 1\right) e^{-\jmath\omega t} dt + \int_{0}^{d/2} \left(1 - \frac{2t}{d}\right) e^{-\jmath\omega t} dt$$

$$= \frac{2}{d}\left[\int_{-d/2}^{0} t e^{-\jmath\omega t} dt - \int_{0}^{d/2} t e^{-\jmath\omega t} dt\right] + \int_{-d/2}^{d/2} e^{-\jmath\omega t} dt$$

$$\int_{-d/2}^{0} t e^{-\jmath\omega t} dt = -\frac{d}{\jmath 2\omega} e^{\jmath\omega d/2} + \frac{1}{\omega^2} - \frac{e^{\jmath\omega d/2}}{\omega^2}$$

$$\int_{0}^{d/2} t e^{-\jmath\omega t} dt = -\frac{d}{\jmath 2\omega} e^{\jmath\omega d/2} - \frac{1}{\omega^2} + \frac{e^{-\jmath\omega d/2}}{\omega^2}$$

$$\int_{-d/2}^{d/2} e^{-\jmath\omega t} dt = \frac{e^{\jmath\omega d/2} - e^{-\jmath\omega d/2}}{\jmath\omega} = \frac{2}{\omega}\sin\frac{\omega d}{2}$$

$$X(\omega) = \frac{2}{d}\left[\frac{2}{\omega^2} - \left(\frac{e^{\jmath\omega d/2} + e^{-\jmath\omega d/2}}{\omega^2}\right) + \left(\frac{-d}{\jmath 2\omega}\right)\left(e^{\jmath\omega d/2} - e^{-\jmath\omega d/2}\right)\right] + \frac{2}{\omega}\sin\frac{\omega d}{2}$$

$$= \frac{4}{d\omega^2}\left[1 - \cos\frac{\omega d}{2}\right] - \frac{2}{\omega}\sin\frac{\omega d}{2} + \frac{2}{\omega}\sin\frac{\omega d}{2}$$

$$= \frac{8}{d\omega^2}\sin^2\frac{\omega d}{4}$$

$$= \frac{d}{2} sinc^2\left(\frac{\omega d}{4\pi}\right) \quad \square$$

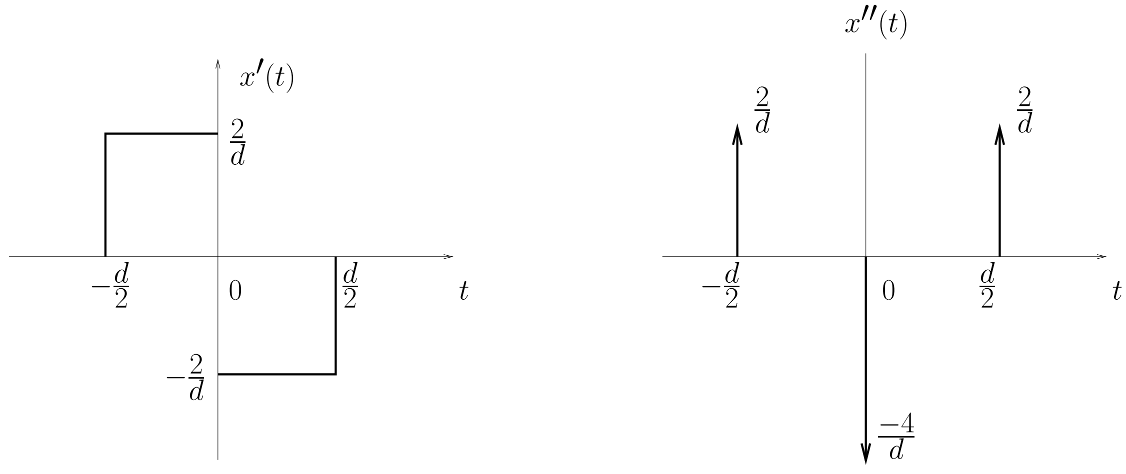


Figure 5: Problem 4

Method 2:

$$y(t) = x''(t) = \frac{2}{d}\delta\left(t + \frac{d}{2}\right) - \frac{4}{d}\delta(t) + \frac{2}{d}\delta\left(t - \frac{d}{2}\right)$$

$$Y(\omega) = \frac{2}{d}e^{-j\omega d/2} - \frac{4}{d} + \frac{2}{d}e^{j\omega d/2}$$

$$= \frac{2}{d}\left[2\cos\frac{\omega d}{2}\right] - \frac{4}{d} = -\frac{4}{d}\left[1 - \cos\frac{\omega d}{2}\right]$$

$$X(\omega) = \frac{1}{j\omega^2}\frac{4}{d}\left[1 - \cos\frac{\omega d}{2}\right] = +\frac{4}{d\omega^2}\left[1 - \cos\frac{\omega d}{2}\right]$$

$$= -\frac{4}{d\omega^2}\left[\cos\frac{\omega d}{2} - 1\right] = +\frac{4}{d\omega^2}\left[+2\sin^2\left(\frac{\omega d}{4}\right)\right]$$

$$= \frac{8}{d\omega^2}\sin^2\left(\frac{\omega d}{4}\right)$$

$$\therefore X(\omega) = \frac{d}{2}sinc^2\left(\frac{\omega d}{4\pi}\right)$$

Method 3:

$$x(t) = \left[rect\left(\frac{t}{d/2}\right) * rect\left(\frac{t}{d/2}\right)\right]\frac{2}{d}$$

We know

$$rect\left(\frac{t}{d}\right) \Longleftrightarrow d\ \text{sinc}\left(\frac{\omega d}{2\pi}\right)$$

$$\Longrightarrow rect\left(\frac{t}{d/2}\right) \Longleftrightarrow \frac{d}{2}\ \text{sinc}\left(\frac{\omega d}{4\pi}\right)$$

$$\Longrightarrow X(\omega) = \left[\frac{d^2}{4} sinc^2\left(\frac{\omega d}{4\pi}\right)\right]\frac{2}{d} = \frac{d}{2} sinc^2\left(\frac{\omega d}{4\pi}\right)$$

5. From the given $x(t)$, $x'(t)$ and $x''(t)$ can be determined to be as shown in the figure below.

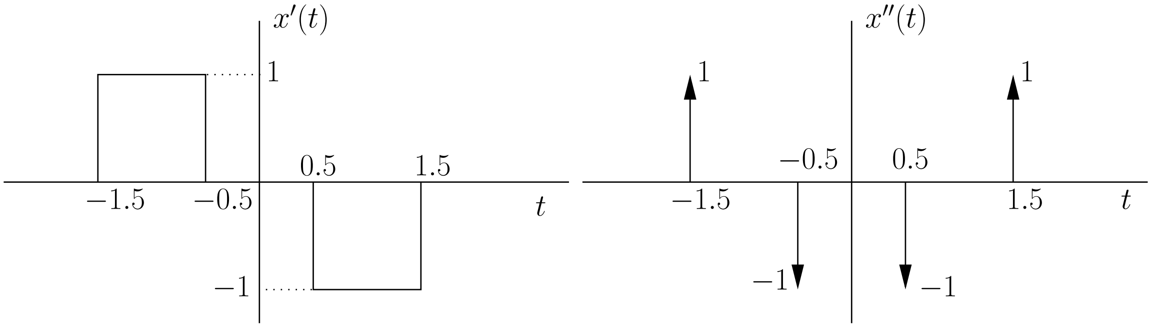


Figure 6: Problem 8

Now $y(t) = x''(t) = \delta(t-1.5) + \delta(t+1.5) - \delta(t-0.5) - \delta(t+0.5)$. Therefore, we have

$$Y(\omega) = e^{-j1.5\omega} + e^{j1.5\omega} - e^{-j0.5\omega} - e^{j0.5\omega} = 2\left[\cos 1.5\omega - \cos 0.5\omega\right].$$

$$X(\omega) = \frac{Y(\omega)}{-\omega^2} = \frac{2}{\omega^2}\left[\cos 1.5\omega - \cos 0.5\omega\right].$$

6. $E_x = \int_{-\infty}^{\infty} x^2(t)dt = \int_0^{\infty} e^{-2at}dt = \frac{1}{2a}$

$$X(\omega) = \frac{1}{a + \jmath\omega}$$

We need to find W such that

$$\frac{1}{2\pi}\int_{-W}^{W}|X(\omega)|^2 d\omega = \frac{0.95}{2a}$$

$$\text{i.e. } \frac{1}{\pi}\int_{0}^{W}\frac{1}{a^2+\omega^2}d\omega = \frac{0.95}{2a}$$

$$\text{i.e. } \frac{1}{\pi a}\tan^{-1}\left(\frac{\omega}{a}\right)\Big|_0^W = \frac{0.95}{2a}$$

$$\Longrightarrow \frac{0.95\pi}{2} = \tan^{-1}\left(\frac{W}{a}\right) \Longrightarrow W = 12.706 \text{ rad/s}$$

7. $x(t) = \dfrac{2a}{t^2+a^2}$

We know $e^{-a|t|} \Longleftrightarrow \frac{2a}{a^2+\omega^2}$, for $a > 0$.

Using the duality property we get

$$\frac{2a}{a^2+t^2} \Longleftrightarrow 2\pi e^{-a|\omega|}$$

$$(\text{if } x(t) \Longleftrightarrow X(\omega), \text{ then } X(t) \Longleftrightarrow 2\pi x(-\omega))$$

Therefore, we have

$$E_x = \int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\pi^2 e^{-2a|\omega|}d\omega = 2\pi\left[2\int_0^{\infty} e^{-2a\omega}\right]$$

$$= 4\pi \left.\frac{e^{-2a\omega}}{-2a}\right|_0^{\infty} = 4\pi\left[0 - \frac{1}{-2a}\right]$$

$$= \frac{2\pi}{a}$$

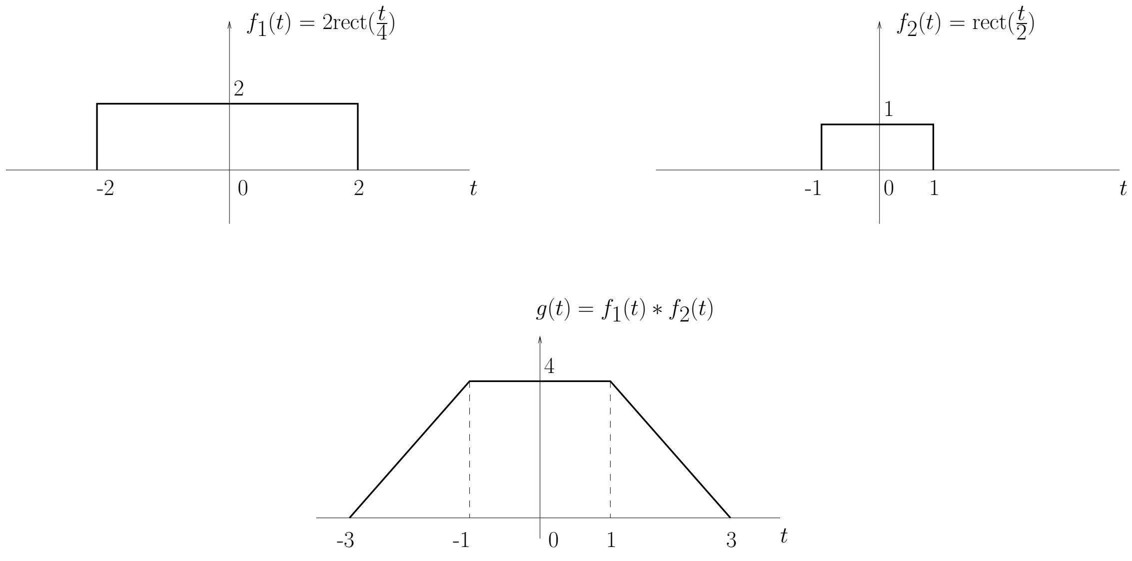
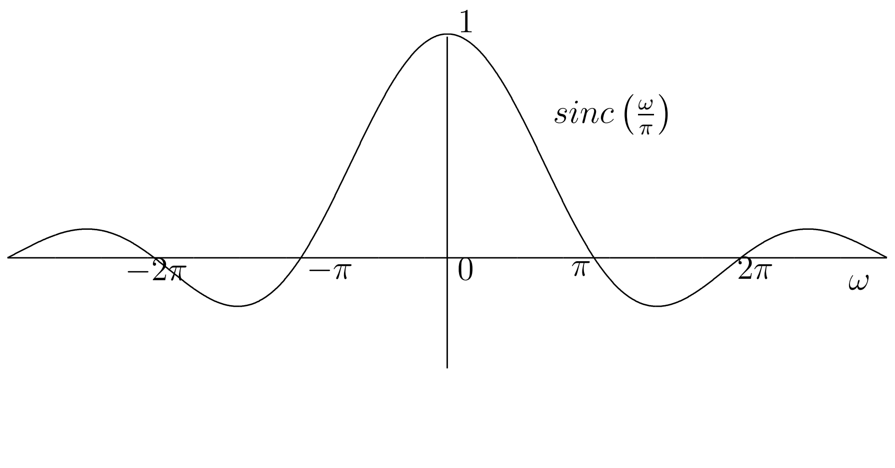


Figure 7: Problem 8

8.

$$G(\omega) = F_1(\omega)F_2(\omega)$$
$$F_1(\omega) = 2\left[4\ \text{sinc}\left(\frac{4\omega}{2\pi}\right)\right] = 8\ \text{sinc}\left(\frac{2\omega}{\pi}\right)$$
$$F_2(\omega) = 2\ \text{sinc}\left(\frac{2\omega}{2\pi}\right) = 2\ \text{sinc}\left(\frac{\omega}{\pi}\right)$$
$$\Longrightarrow G(\omega) = 16\ \text{sinc}\left(\frac{\omega}{\pi}\right)\ \text{sinc}\left(\frac{2\omega}{\pi}\right)$$



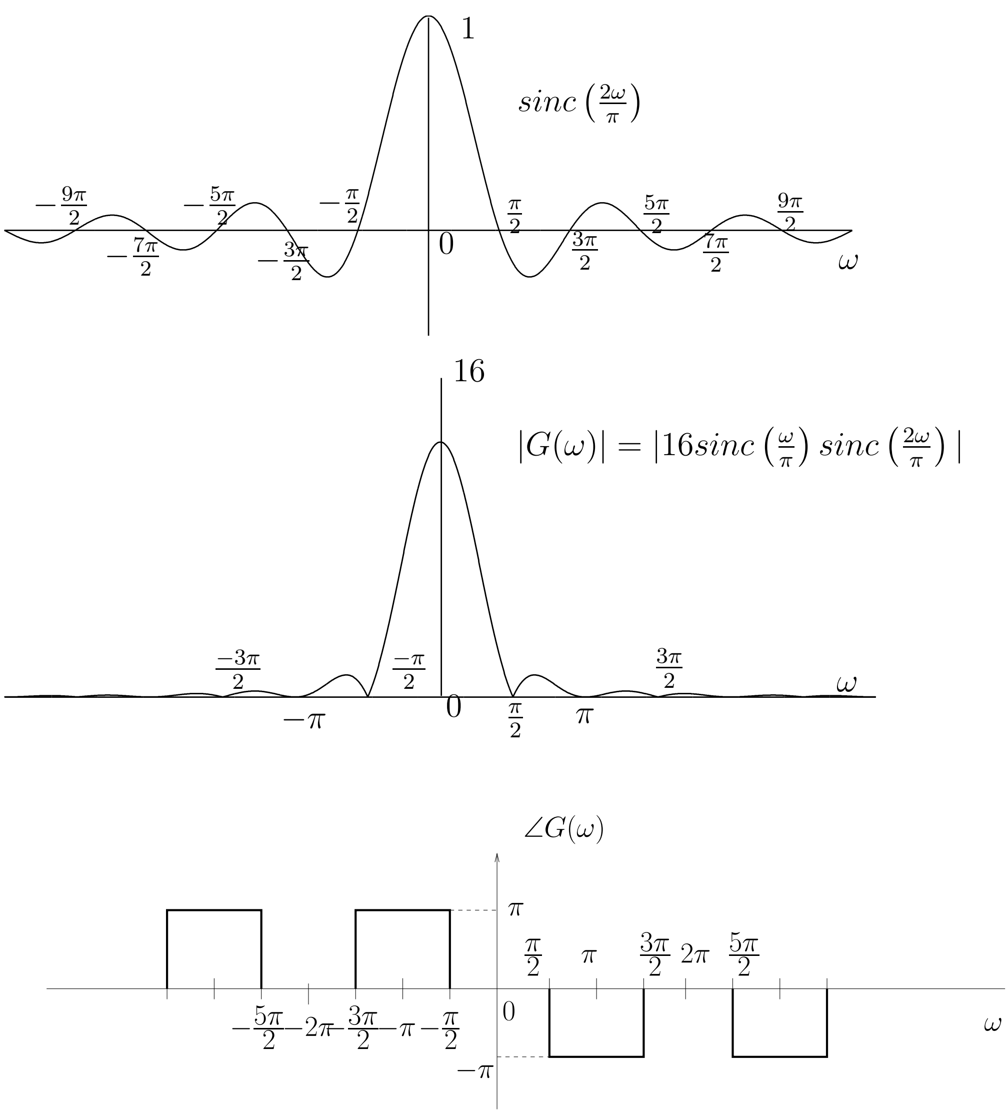


Figure 8: Problem 8

9. Let $x_1(t)$ be bandlimited to B_1 Hz & let $x_2(t)$ be bandlimited to B_2 Hz.

$$y(t) = x_1(t)x_2(t) \Longleftrightarrow \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\alpha)X_2(\omega-\alpha)d\alpha$$

If $\omega + 2\pi B_2 < -2\pi B_1$, $Y(\omega) = 0$,

i.e. if $\omega < -2\pi(B_1 + B_2)$, $Y(\omega) = 0$

If $\omega - 2\pi B_2 > 2\pi B_1$, $Y(\omega) = 0$

i.e. if $\omega > 2\pi(B_1 + B_2)$, $Y(\omega) = 0$

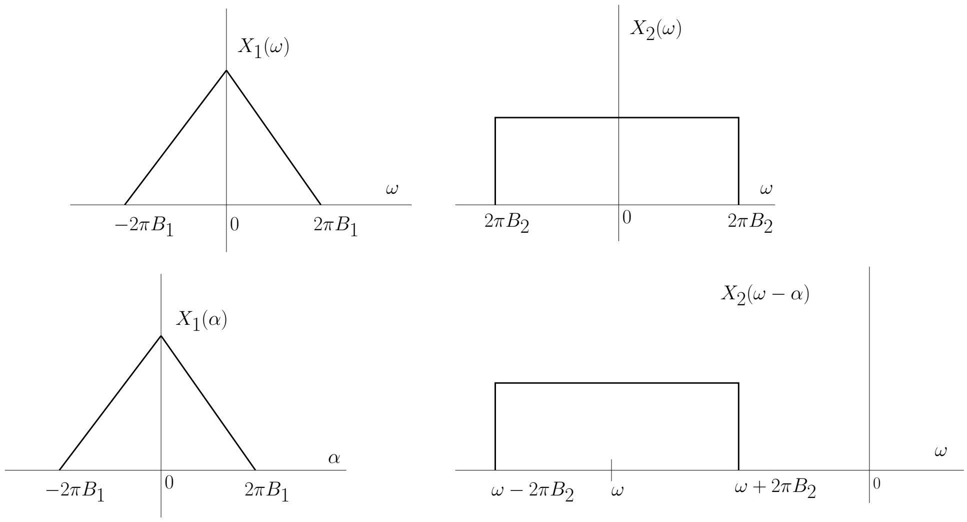


Figure 9: Problem 9

i.e. $y(t)$ is bandlimited to $(B_1 + B_2)$ Hz.

Repeatedly using the above result we get $x^n(t)$ is bandlimited to nB Hz.

10.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-\jmath\omega t}dt$$

$$\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty} x(t)[-\jmath te^{-\jmath\omega t}]dt$$

$$= \int_{\infty}^{\infty} [-\jmath tx(t)]e^{-\jmath\omega t}dt$$

Therefore, we have

$$-\jmath tx(t) \Longleftrightarrow \frac{X(\omega)}{d\omega}$$

$$e^{-at}u(t) \Longleftrightarrow \frac{1}{a+\jmath\omega}$$

$$-\jmath te^{-at}u(t) \Longleftrightarrow \frac{d}{d\omega}\left[\frac{1}{a+\jmath\omega}\right] = \frac{-1}{(a+\jmath\omega)^2}(\jmath)$$

$$te^{-at}u(t) \Longleftrightarrow \frac{1}{(a+\jmath\omega)^2}$$

11.

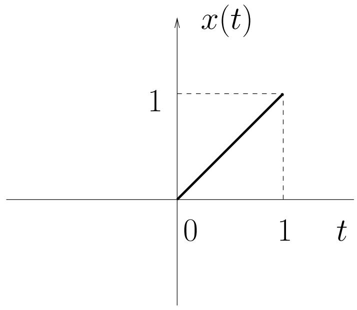


Figure 10: Problem 11

Now,

$$X(\omega) = \int_0^1 te^{-\jmath\omega t}dt = \int_0^1 td\left(\frac{e^{-\jmath\omega t}}{-\jmath\omega}\right)$$

$$= \frac{te^{-\jmath\omega t}}{-\jmath\omega}\bigg|_0^1 - \frac{e^{-\jmath\omega t}}{(-\jmath\omega)^2}\bigg|_0^1$$

$$= \frac{e^{-\jmath\omega}}{-\jmath\omega} - \left[\frac{e^{-\jmath\omega} - 1}{-\omega^2}\right]$$

$$= \frac{+\jmath\omega e^{-\jmath\omega} + e^{-\jmath\omega} - 1}{\omega^2} = \frac{1}{\omega^2}[e^{-\jmath\omega} + \jmath\omega e^{-\jmath\omega} - 1]$$

We observe that

$$x_1(t) = x(-t+1) + x(t+1)$$

$$\therefore X_1(\omega) = X(-\omega)e^{-\jmath\omega} + X(\omega)e^{\jmath\omega}$$

$$= \frac{1}{\omega^2}[(e^{\jmath\omega} - \jmath\omega e^{\jmath\omega} - 1)e^{-\jmath\omega} + (e^{-\jmath\omega} + \jmath\omega e^{-\jmath\omega} - 1)e^{\jmath\omega}]$$

$$= \frac{1}{\omega^2}[2 - 2\cos\omega] = \frac{4}{\omega^2}\sin^2\left(\frac{\omega}{2}\right)$$

$$= \text{ sinc}^2\left(\frac{\omega}{2\pi}\right)$$

Again, notice that

$$x_2(t) = x(-t+1) + x(t) + x(t+1) + x(-t)$$

$$\therefore X_2(\omega) = X(-\omega)e^{-\jmath\omega} + X(\omega)e^{\jmath\omega} + X(\omega) + X(-\omega)$$

$$= \frac{1}{\omega^2}[\jmath\omega(e^{-\jmath\omega} - e^{\jmath\omega})] = \frac{-\jmath}{\omega}(2\jmath\sin\omega)$$

$$= 2\frac{\sin\omega}{\omega} = 2\ \mathrm{sinc}\left(\frac{\omega}{\pi}\right)$$