EC204: Networks & Systems
Solution to Problem Set 10

1. Here,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

Hence, the transfer function $\mathbf{H}(s)$ is given by

$$\begin{aligned} \mathbf{H}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{s+4}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{s+4}{s+2} & \frac{1}{s+2} \\ \frac{2(s-2)}{(s+1)(s+2)} & \frac{s^2+5s+2}{(s+1)(s+2)} \end{bmatrix} \end{aligned}$$

and the zero-state response is

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s)$$

The transfer function that relates the output y_3 to the input u_2 is $H_{32}(s)$, where

$$H_{32}(s) = \frac{s^2+5s+2}{(s+1)(s+2)}$$

2. We have

$$(s\mathbf{I}-\mathbf{A}) = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -12 & 2/3 \\ -36 & -1 \end{bmatrix} = \begin{bmatrix} s+12 & -2/3 \\ 36 & s+1 \end{bmatrix}$$

and

$$(s\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} \dfrac{s+1}{(s+4)(s+9)} & \dfrac{2/3}{(s+4)(s+9)} \\ \dfrac{-36}{(s+4)(s+9)} & \dfrac{s+12}{(s+4)(s+9)} \end{bmatrix}$$

Now, $\mathbf{x}(0)$ is given as

$$\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Also, $U(s) = 1/s$, and

$$\mathbf{B}\mathbf{U}(s) = \begin{bmatrix} \dfrac{1}{3} \\ 1 \end{bmatrix} \frac{1}{s} = \begin{bmatrix} \dfrac{1}{3s} \\ \dfrac{1}{s} \end{bmatrix}$$

Therefore

$$\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s) = \begin{bmatrix} 2 + \dfrac{1}{3s} \\ 1 + \dfrac{1}{s} \end{bmatrix} = \begin{bmatrix} \dfrac{6s+1}{3s} \\ \dfrac{s+1}{s} \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{X}(s) &= (s\mathbf{I}-\mathbf{A})^{-1}\left[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)\right] \\ &= \begin{bmatrix} \dfrac{s+1}{(s+4)(s+9)} & \dfrac{2/3}{(s+4)(s+9)} \\ \dfrac{-36}{(s+4)(s+9)} & \dfrac{s+12}{(s+4)(s+9)} \end{bmatrix} \begin{bmatrix} \dfrac{6s+1}{3s} \\ \dfrac{s+1}{s} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{2s^2+3s+1}{s(s+4)(s+9)} \\ \dfrac{s-59}{(s+4)(s+9)} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{1/36}{s} - \dfrac{21/20}{s+4} + \dfrac{136/45}{s+9} \\ \dfrac{-63/5}{s+4} + \dfrac{68/5}{s+9} \end{bmatrix} \end{aligned}$$

The inverse Laplace transform of this equation yields

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\dfrac{1}{36} - \dfrac{21}{20}e^{-4t} + \dfrac{136}{45}e^{-9t}\right)u(t) \\ \left(-\dfrac{63}{5}e^{-4t} + \dfrac{68}{5}e^{-9t}\right)u(t) \end{bmatrix}$$

3. (a) The state and output equations are:

$$\begin{bmatrix} \dot{v_C} \\ \dot{i_L} \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -1/2 & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

$$i = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

(b) The initial conditions are: $v_C(0^-) = 3\ V$ and $i_L(0^-) = 2\ A$. The zero input solution is $3e^{-t}u(t) + te^{-t}u(t)$. The zero state solution is $2u(t) - 3e^{-t}u(t) - te^{-t}u(t)$. The total solution is $2u(t)$.