Gaussian many-to-one channels
Some sum capacity results[1]

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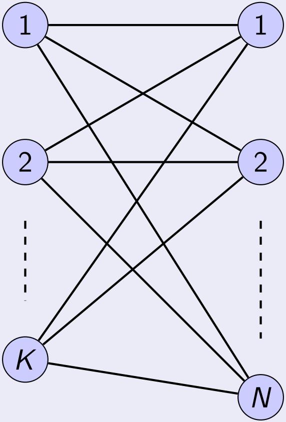
Joint work with Ranga Prasad and A. Chockalingam (IISc)

[1] R. Prasad, S. Bhashyam, A. Chockalingam, "On the Gaussian Many-to-One X Channel," IEEE Transactions on Information Theory, vol. 62, no. 1, pp. 244-259, January 2016.

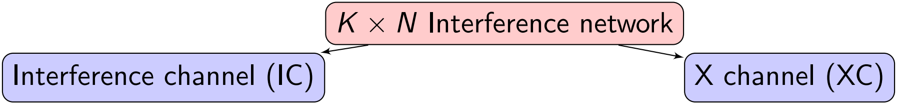
Single-hop Interference Network [Carleial '78]



- K transmitters, N receivers, single-hop
- Possible message from each transmitter to each subset of receivers

$$K(2^N - 1) \text{ possible messages}$$

Single-hop interference networks: Two special cases

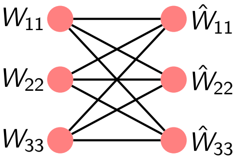


K-user IC

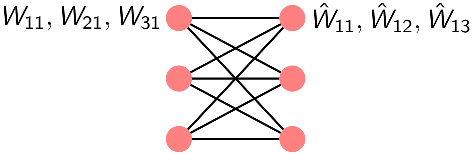
- K distinct Tx-Rx pairs, K messages
- $K(K-1)$ interfering links

K $\times$ N XC

- Message for each link
- KN messages



3-user IC



3×3 XC

Gaussian interference channel: Some known results

2-user IC

- Capacity
  - Strong interference [Carleial75, HanKob81, Sato81]
- Sum capacity
  - Mixed interference [MotKha09]
  - Noisy interference [AnnVee09, MotKha09, ShaKraChe09]

K-user IC

- Sum capacity: Noisy interference [ShaKraChe09, AnnVee09]
- Capacity region: Symmetric very strong interference [SriJafVisJaf08]

K-user Many-to-one IC

- Sum capacity: Noisy interference [CadJaf09, AnnVee09]
- Capacity region: Symmetric very strong interference [ZhuGas15]
- Special case ($K = 2$): One-sided IC / Z-IC [Sason04]
- This talk (Sum capacity)

Gaussian X channel: Some known results

2×2 XC

- Sum capacity: Noisy interference [HuaCadJaf12]

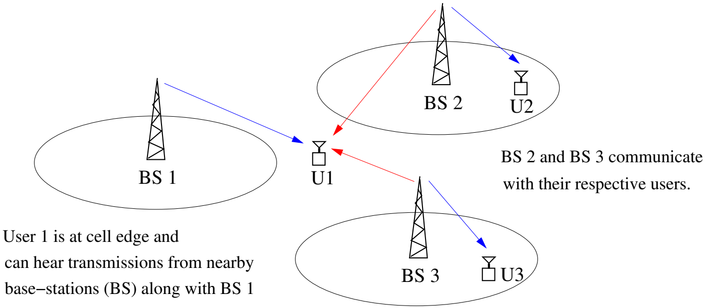
$K \times K$ Many-to-one XC

- Special case ($K = 2$): ZC
  - Sum capacity: Weak interference [LiuUlu04]
  - Sum capacity: Strong interference [ChoMotGar07]
  - Capacity region: Moderately strong interference [ChoMotGar07]
- This talk (Sum capacity)

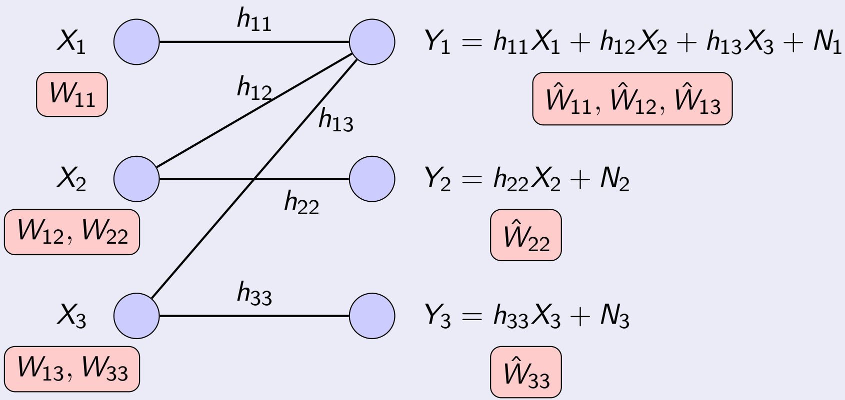
Gaussian many-to-one channels

Many-to-one Channels: Motivation

- Captures some essential features (interference), easier for analysis
- Results can be used to find bounds for more general topologies
  - Bounds for 2-user IC using one-sided IC
- Approximation to a possible real scenario



3×3 Gaussian many-to-one X channel

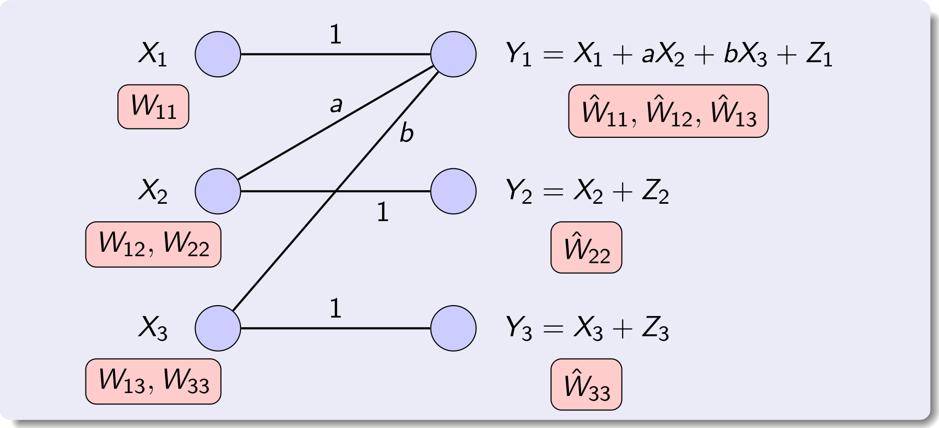


Capacity region (5-dimensional) not easy to characterize

- One flow on each link (R_{ij}: Rate from Tx j to Rx i)
- $\mathcal{C}$ = Set of all achievable $\mathbf{R} = (R_{11}, R_{22}, R_{12}, R_{33}, R_{13})$
- This talk: sum capacity

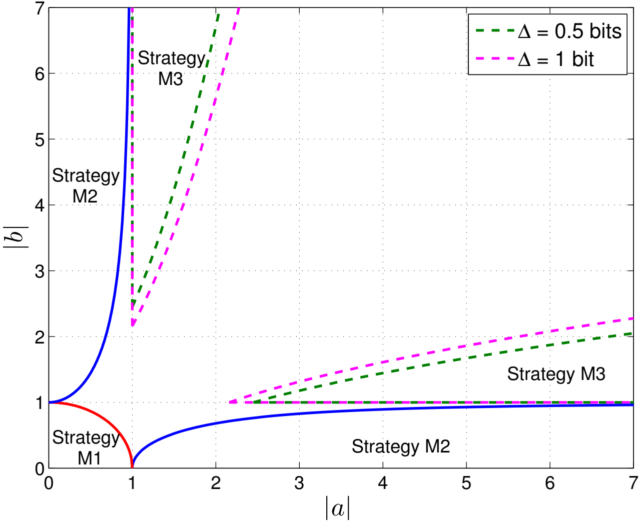
Channel in standard form

Reduce the number of parameters required



- $\mathcal{C}(\mathbf{P}', \mathbf{h}, \mathbf{N}) = \mathcal{C}_{standard}(\mathbf{P}, a, b)$
- Z_i IID $\sim N(0, 1)$, $\mathbf{P}$, $\mathbf{P}'$: power constraints, $\mathbf{N}$: noise variance vector

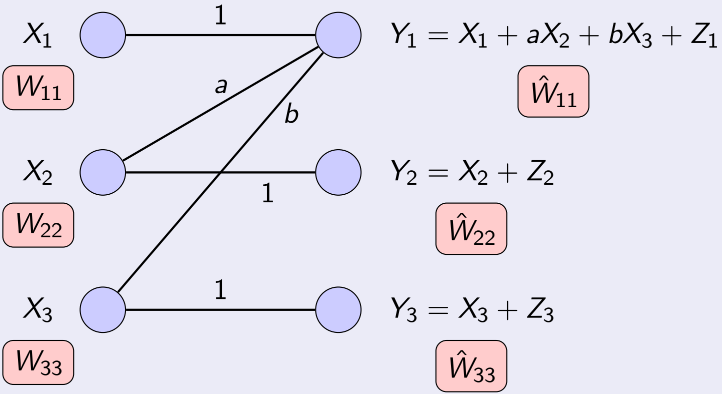
Result for the 3 $\times$ 3 many-to-one XC



$P_1 = P_2 = P_3 = 0\text{dB}$

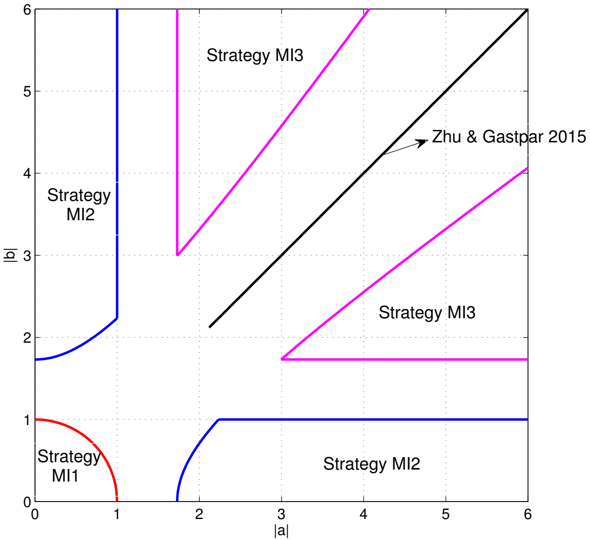
Many-to-one IC

A special case of the many-to-one XC



- Sum capacity: Noisy interference Annapureddy & Veeravalli 2009, Cadambe & Jafar 2009
- Capacity: Symmetric very strong interference Zhu & Gastpar 2015
- Capacity within a constant gap Bresler, Parekh & Tse 2010, Jovicic, Wang, & Viswanath 2010

Result for the 3 × 3 many-to-one IC



$P_1 = P_2 = P_3 = 3\text{dB}$

Rest of this talk

3×3 Many-to-one XC

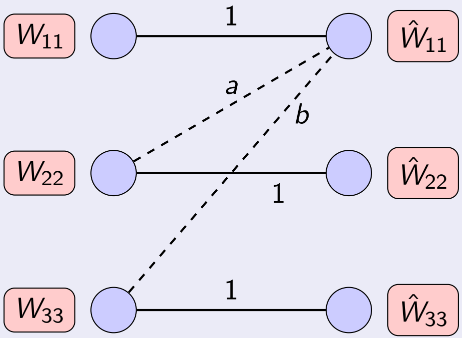
- Transmission strategies for the many-to-one XC
  - Treat interference from a subset of transmitters as noise
  - Use of Gaussian codebooks
- Conditions for sum rate optimality

Extensions to $K \times K$ Many-to-one XC

$K \times K$ Many-to-one IC

Gaussian 3×3 many-to-one XC

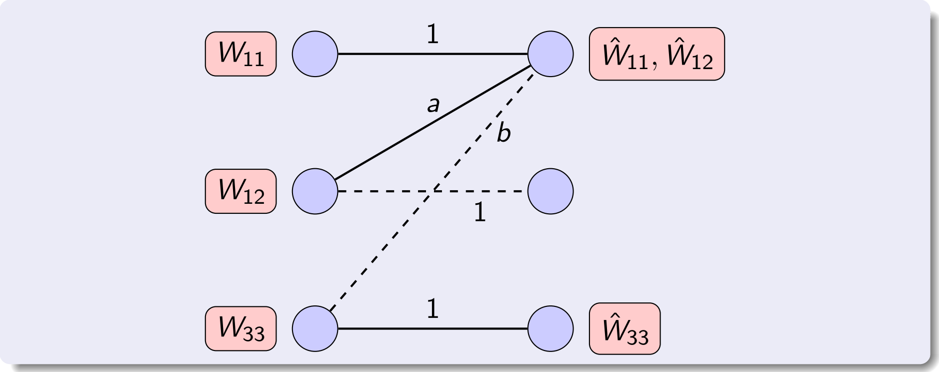
Strategy M1: Treating Interference as Noise (TIN)



Achieved sum-rate

$$R_{sum} = \frac{1}{2}\log_2\left(1 + \frac{P_1}{a^2P_2 + b^2P_3 + 1}\right) + \frac{1}{2}\log_2(1+P_2) + \frac{1}{2}\log_2(1+P_3)$$

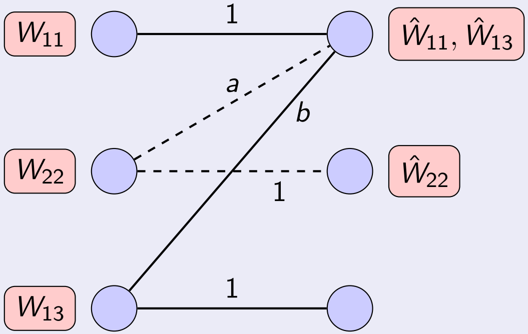
Strategy M2



Achieved sum-rate

$$R_{sum} = \frac{1}{2}\log_2\left(1 + \frac{P_1 + a^2 P_2}{b^2 P_3 + 1}\right) + \frac{1}{2}\log_2(1 + P_3)$$

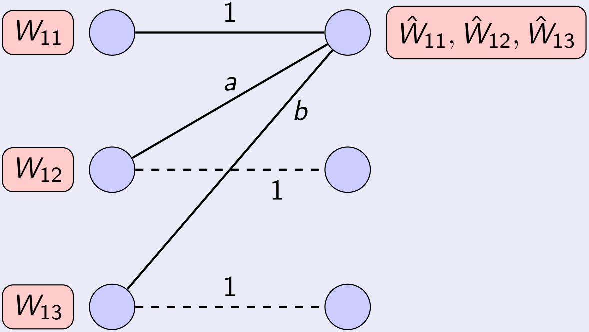
Strategy M2



Achieved sum-rate

$$R_{sum} = \frac{1}{2}\log_2\left(1 + \frac{P_1 + b^2 P_3}{a^2 P_2 + 1}\right) + \frac{1}{2}\log_2(1 + P_2)$$

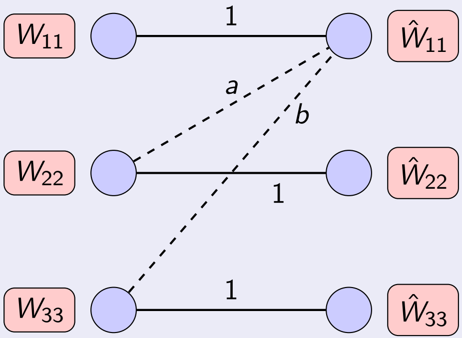
Strategy M3



Achieved sum-rate

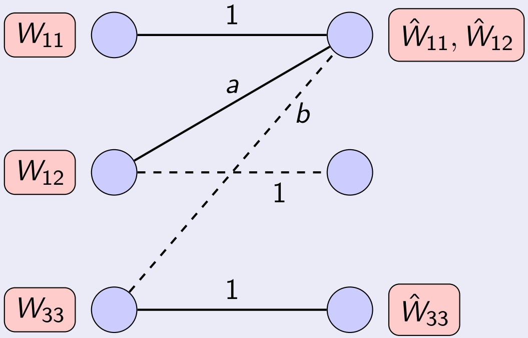
$$R_{sum} = \frac{1}{2}\log_2\left(1 + P_1 + a^2 P_2 + b^2 P_3\right)$$

Sum-rate optimality of Strategy M1 (TIN)



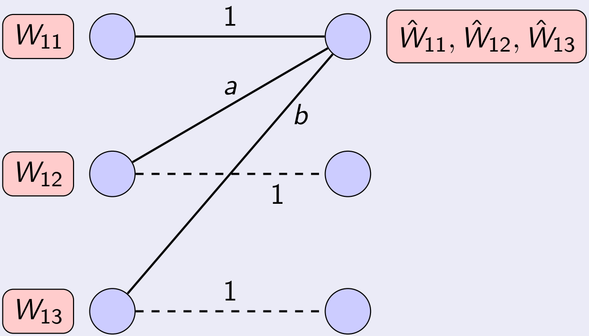
Strategy M1 achieves sum capacity if $a^2 + b^2 \leq 1$

Sum-rate optimality of Strategy M2



Strategy M2 achieves sum capacity if $b^2 < 1$ and $a^2 \geq \frac{(1+b^2 P_3)^2}{1-b^2}$

Approximate sum-rate optimality of Strategy M3



Strategy M3 achieves rates within

$$\frac{1}{2}\log_2\left(\frac{1-(1+b^2P_3)^{-1}\rho^2}{1-\rho^2}\right) \text{ bits}$$

of sum capacity if $b^2 \geq 1$ and $a^2 \geq \frac{(1+b^2P_3)^2}{\rho^2}$

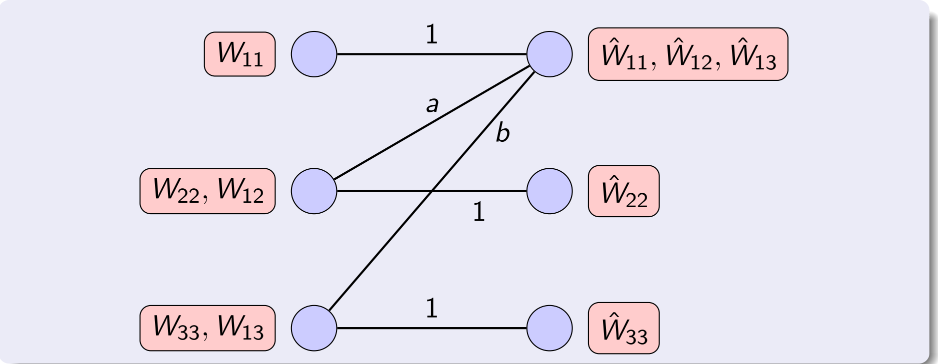
Sum-rate optimality proofs: Outline

Need an upper bound that matches achievable sum-rate

Upper bound using

- Fano's inequality
- Degraded receivers
- Worst-case additive noise result, Entropy-Power inequality (EPI)
- Genie-aided channel/Channel with side information (M2 & M3)

Degraded receivers

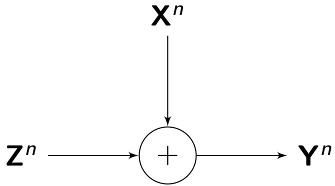


- If $a^2 \leq 1$, Rx 1 is a degraded version of Rx 2 w.r.t. W_{12}

$$H(W_{12}|\mathbf{y}_2^n) \leq H(W_{12}|\mathbf{y}_1^n) \leq n\epsilon_n$$

- If $b^2 \leq 1$, Rx 1 is a degraded version of Rx 3 w.r.t. W_{13}

Worst-case additive noise



- $\mathbf{Z}^n \sim \mathcal{N}(\mathbf{0}, \Sigma_Z)$ IID
- $\mathbf{X}^n$: average covariance constraint Σ_X

Worst case noise result (Diggavi & Cover 01, Annapureddy & Veeravalli 09)

$$h(\mathbf{X}^n) - h(\mathbf{X}^n + \mathbf{Z}^n) \leq nh(\mathbf{X}_G) - nh(\mathbf{X}_G + \mathbf{Z}),$$

where $\mathbf{X}_G \sim \mathcal{N}(\mathbf{0}, \Sigma_X)$.

Another result[2]

$$\sum_{i=1}^{K} h(X_i^n + Z_i^n) \quad - \quad h\Big(\sum_{i=1}^{K} c_i\, X_i^n + Z_1^n\Big)$$

$$\leq \quad n \sum_{i=1}^{K} h(X_{iG} + Z_i) - nh\Big(\sum_{i=1}^{K} c_i\, X_{iG} + Z_1\Big),$$

$$\text{if} \quad \sum_{i=1}^{K} c_i^2 \leq \sigma^2$$

- X_i^n with power constraint $\sum_{j=1}^{n} \mathbb{E}[(X_{ij}^2] \leq nP_i$
- Z_1^n vector with IID $\mathcal{N}(0, \sigma^2)$ components
- Z_i^n, $i \neq 1$ vector with IID $\mathcal{N}(0, 1)$ components
- X_i^n are independent of Z_i^n
- $X_{iG} \sim \mathcal{N}(0, P_i)$

[2] Lemma 5 from Annapureddy & Veeravalli 2009 in different form

Proof of sum-rate optimality of Strategy M1 (1)

Let S denote any achievable sum-rate. Want to show

$$S \leq I(x_{1G}; y_{1G}) + I(x_{2G}; y_{2G}) + I(x_{3G}; y_{3G}).$$

$$\begin{aligned}
nS &\leq H(W_{11}) + H(W_{12}, W_{22}) + H(W_{13}, W_{33}) \\
&= I(W_{11}\,;\,\mathbf{y}_1^n) + \sum_{i=2}^{3} I(W_{1i}, W_{ii}\,;\,\mathbf{y}_i^n) \\
&\quad + H(W_{11}\,|\,\mathbf{y}_1^n) + \sum_{i=2}^{3} H(W_{1i}, W_{ii}\,|\,\mathbf{y}_i^n) \\
&\leq I(\mathbf{x}_1^n\,;\,\mathbf{y}_1^n) + \sum_{i=2}^{3} I(\mathbf{x}_i^n\,;\,\mathbf{y}_i^n) \\
&\quad + H(W_{11}\,|\,\mathbf{y}_1^n) + \sum_{i=2}^{3} H(W_{1i}, W_{ii}\,|\,\mathbf{y}_i^n)
\end{aligned}$$

Proof of sum-rate optimality of Strategy M1 (2)

$$
\begin{aligned}
nS &\leq I(\mathbf{x}_1^n;\mathbf{y}_1^n) + \sum_{i=2}^{3} I(\mathbf{x}_i^n;\mathbf{y}_i^n) + H(W_{11}\,|\,\mathbf{y}_1^n) + \sum_{i=2}^{3} H(W_{1i}, W_{ii}\,|\,\mathbf{y}_i^n) \\
&\overset{(a)}{\leq} h(\mathbf{y}_1^n) - h(a\mathbf{x}_2^n + b\mathbf{x}_3^n + \mathbf{n}_1^n) + h(\mathbf{x}_2^n + \mathbf{n}_2^n) - h(\mathbf{n}_2^n) + h(\mathbf{x}_3^n + \mathbf{n}_3^n) \\
&\quad - h(\mathbf{n}_3^n) + 5\epsilon_n \\
&\overset{(b)}{\leq} nh(y_{1G}) - nh(ax_{2G} + bx_{3G} + n_1) + nh(x_{2G} + n_2) + nh(x_{3G} + n_3) \\
&\quad - nh(n_2) - nh(n_3) + 5\epsilon_n \\
&= nI(x_{1G}; y_{1G}) + nI(x_{2G}; y_{2G}) + nI(x_{3G}; y_{3G}) + 5\epsilon_n,
\end{aligned}
$$

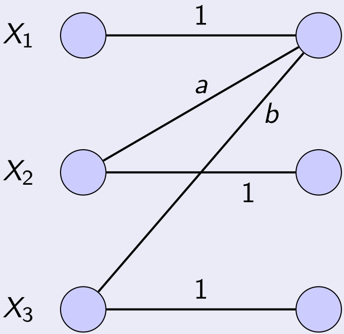
(a): Fano's inequality, $a^2 \leq 1$ and $b^2 \leq 1$

(b): Generalized form of worst-case noise result, $a^2 + b^2 \leq 1$

Proof of sum-rate optimality of Strategy M2 (1)

Want to show $S \leq I(x_{1G}, x_{2G}; y_{1G}) + I(x_{3G}; y_{3G})$.

$S_1 = X_1 + aX_2 + \eta N_1$



$Y_1 = X_1 + aX_2 + bX_3 + Z_1$

$Y_2 = X_2 + Z_2$

$Y_3 = X_3 + Z_3$

- Show $S \leq I(x_{1G}, x_{2G}; y_{1G}, s_{1G}) + I(x_{3G}; y_{3G})$
- $E[N_1 Z_1] = \rho$, $\eta > 0$ chosen later

Proof of sum-rate optimality of Strategy M2 (2)

$$
\begin{aligned}
nS &\leq H(W_{11}, W_{12}, W_{22}) + H(W_{13}, W_{33}) \\
&= I(W_{11}, W_{12}, W_{22}\,;\, \mathbf{y}_1^n, \mathbf{s}_1^n) + H(W_{11} \,|\, \mathbf{y}_1^n, \mathbf{s}_1^n) + H(W_{12} \,|\, \mathbf{y}_1^n, \mathbf{s}_1^n, \mathbf{x}_1^n) \\
&\quad + H(W_{22} \,|\, \mathbf{y}_1^n, \mathbf{s}_1^n, \mathbf{x}_1^n, W_{12}) + I(W_{13}, W_{33}; \mathbf{y}_3^n) + H(W_{13}|\mathbf{y}_3^n) \\
&\quad + H(W_{33}|\mathbf{y}_3^n, W_{13}) \\
&\leq I(\mathbf{x}_1^n, \mathbf{x}_2^n\,;\, \mathbf{y}_1^n, \mathbf{s}_1^n) + H(W_{11} \,|\, \mathbf{y}_1^n) + H(W_{12} \,|\, \mathbf{y}_1^n) \\
&\quad + H(W_{22} \,|\, \mathbf{s}_1^n, \mathbf{x}_1^n) + I(\mathbf{x}_3^n\,;\, \mathbf{y}_3^n) + H(W_{13} \,|\, \mathbf{y}_3^n) + H(W_{33} \,|\, \mathbf{y}_3^n), \\
&\overset{(a)}{\leq} I(\mathbf{x}_1^n, \mathbf{x}_2^n\,;\, \mathbf{y}_1^n, \mathbf{s}_1^n) + I(\mathbf{x}_3^n\,;\, \mathbf{y}_3^n) + 5n\epsilon_n \qquad (1)
\end{aligned}
$$

(a): $\eta^2 \leq a^2$ and $b^2 \leq 1$

Proof of sum-rate optimality of Strategy M2 (3)

$$
\begin{aligned}
nS &\leq I(\mathbf{x}_1^n, \mathbf{x}_2^n \,;\, \mathbf{y}_1^n,\, \mathbf{s}_1^n) + I(\mathbf{x}_3^n \,;\, \mathbf{y}_3^n) + 5n\epsilon_n \\
&= I(\mathbf{x}_1^n, \mathbf{x}_2^n; \mathbf{s}_1^n) + I(\mathbf{x}_1^n, \mathbf{x}_2^n; \mathbf{y}_1^n \,|\, \mathbf{s}_1^n) + I(\mathbf{x}_3^n; \mathbf{y}_3^n) + 5n\epsilon_n \\
&= h(\mathbf{s}_1^n) - h(\mathbf{s}_1^n \,|\, \mathbf{x}_1^n, \mathbf{x}_2^n) + h(\mathbf{y}_1^n \,|\, \mathbf{s}_1^n) \\
&\quad - h(\mathbf{y}_1^n \,|\, \mathbf{s}_1^n, \mathbf{x}_1^n, \mathbf{x}_2^n) + h(\mathbf{y}_3^n) - h(\mathbf{y}_3^n \,|\, \mathbf{x}_3^n) + 5n\epsilon_n \\
&\leq nh(s_{1G}) - nh(\eta\, z_1) + nh(y_{1G} \,|\, s_{1G}) \\
&\quad \boxed{- h(b\,\mathbf{x}_3^n + \tilde{\mathbf{n}}_1^n) + h(\mathbf{x}_3^n + \mathbf{n}_3^n)} - nh(n_3) + 5n\epsilon_n \\
&\overset{(b)}{\leq} nh(s_{1G}) - nh(\eta\, z_1) + nh(y_{1G} \,|\, s_{1G}) \\
&\quad \boxed{- nh(b\,x_{3G} + \tilde{n}_1) + nh(x_{3G} + n_3)} - nh(n_3) + 5n\epsilon_n \\
&= n\,I(x_{1G}, x_{2G} \,;\, y_{1G},\, s_{1G}) + n\,I(x_{3G} \,;\, y_{3G}) + 5n\epsilon_n,
\end{aligned}
$$

(b): $b^2 \leq 1 - \rho^2$

Proof of sum-rate optimality of Strategy M2 (4)

Choose

$$\eta\rho = 1 \; + \; b^2 P_3$$

to get

$$I(x_{1G}, x_{2G}\,;\, y_{1G},\, s_{1G}) = I(x_{1G}, x_{2G}\,;\, y_{1G})$$

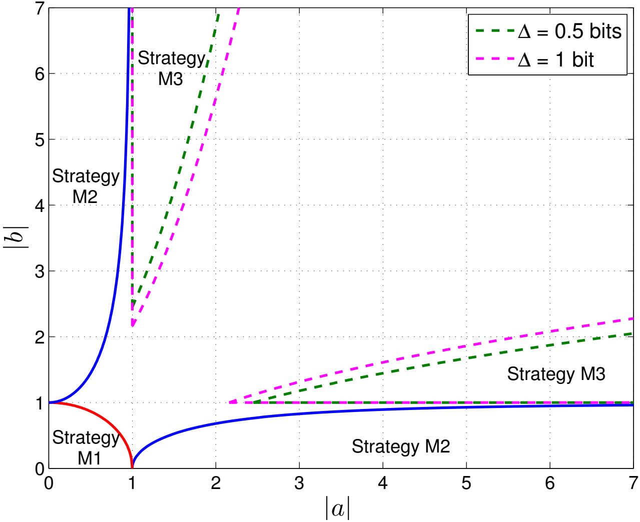
Then, choose

$$\rho^2 = 1 - b^2$$

to get the final result

$$b^2 < 1 \quad \text{and} \quad a^2 \geq \frac{(1 + b^2 P_3)^2}{1 - b^2}$$

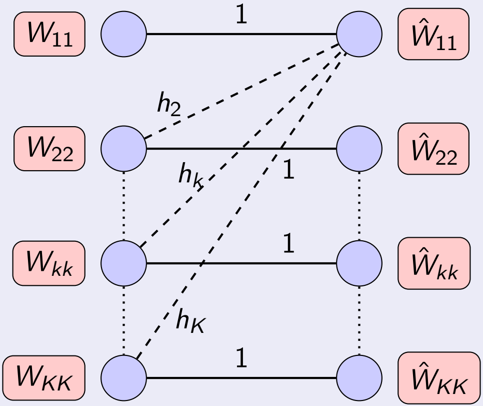
Back to the numerical result



- $P_1 = P_2 = P_3 = 0$ dB

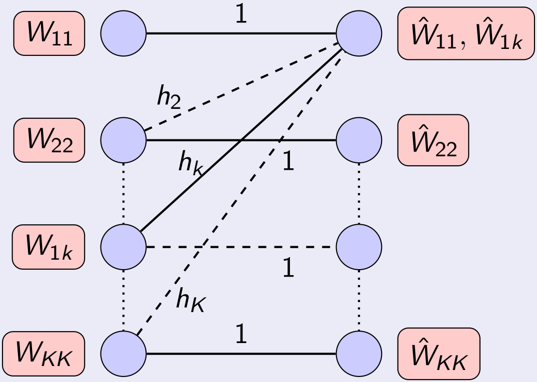
Gaussian $K \times K$ many-to-one XC

Strategy M1 for the $K \times K$ many-to-one XC



Strategy M1 achieves sum capacity if $\sum_{j=2}^{K} h_j^2 < 1$

Strategy M2 for the $K \times K$ many-to-one XC

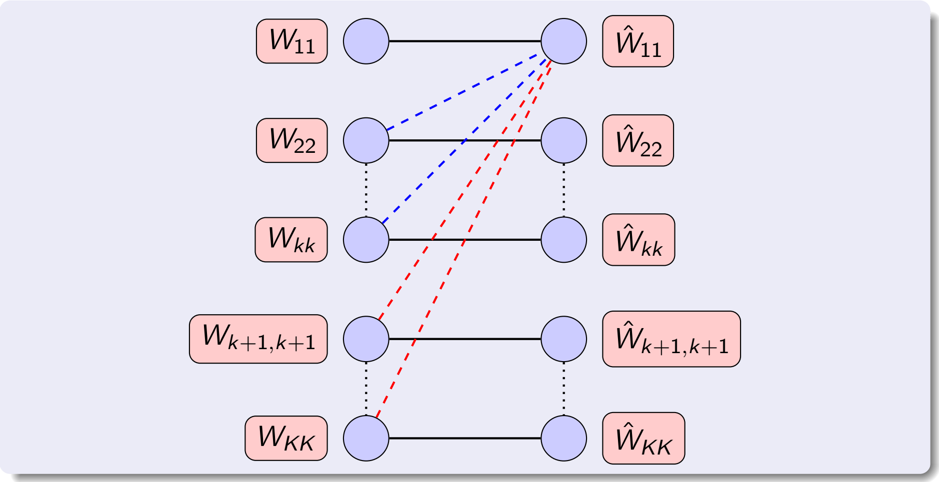


Strategy M2 achieves sum capacity if

$$\sum_{j=2, j\neq k}^{K} h_j^2 < 1 \text{ and } h_k^2 \geq \frac{(1 + \sum_{j=2}^{K} h_j^2 P_j)^2}{1 - \sum_{j=2, j\neq k}^{K} h_j^2}$$

Gaussian K-user many-to-one IC

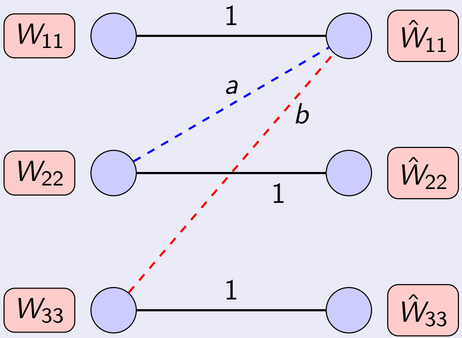
$K \times K$ many-to-one IC



Strategies MIk for $k = 1, 2, \ldots, K$

- Decode interference from transmitters 2 to k (for $k \geq 2$)
- Treat interference from transmitters $k+1$ to K as noise

Strategy MI2 for the 3 $\times$ 3 many-to-one IC (1)



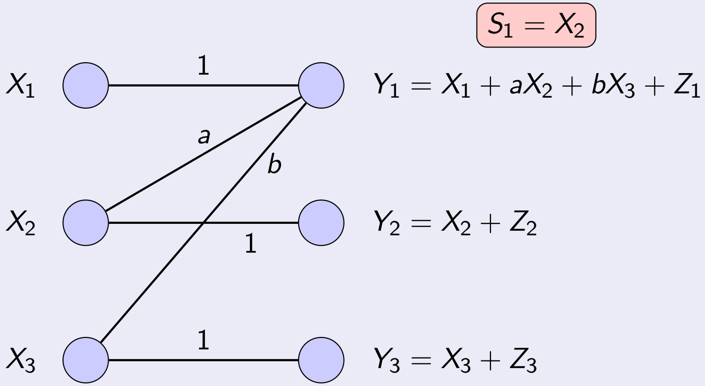
Achieved sum-rate

$$R_{sum} = \frac{1}{2}\log_2\left(1 + \frac{P_1}{b^2 P_3 + 1}\right) + \frac{1}{2}\log_2(1 + P_2) + \frac{1}{2}\log_2(1 + P_3)$$

Required condition: $a^2 \geq 1 + P_1 + b^2 P_3$

Strategy MI2 for the 3 $\times$ 3 many-to-one IC (2)

Want to show $S \leq I(x_{1G}; y_{1G}|x_{2G}) + I(x_{2G}; y_{2G}) + I(x_{3G}; y_{3G})$.

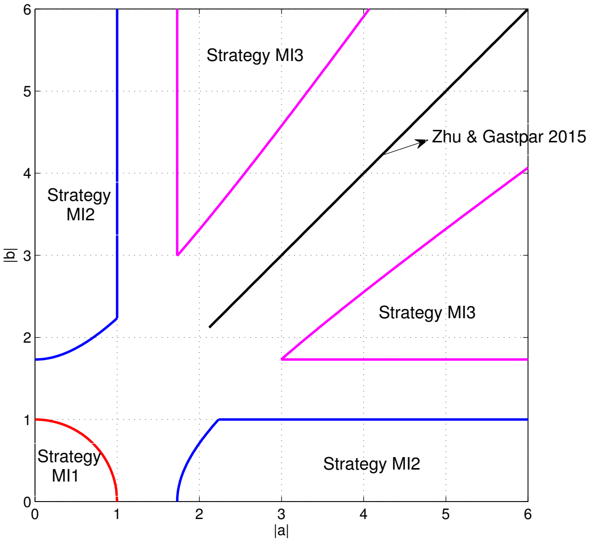


- Need $b^2 \leq 1$

Summary of conditions for 3 $\times$ 3 many-to-one IC

Strategy	Channel conditions
$\mathcal{MI}1$	$a^2 + b^2 \leq 1$
$\mathcal{MI}2$	(i) $a^2 \geq 1 + P_1 + b^2 P_3, \;\; b^2 \leq 1$
	(ii) $b^2 \geq 1 + P_1 + a^2 P_2, \;\; a^2 \leq 1$
$\mathcal{MI}3$	(i) $a^2 \geq 1 + P_1 + b^2 P_3, \;\; b^2 \geq 1 + P_1$
	(ii) $b^2 \geq 1 + P_1 + a^2 P_2, \;\; a^2 \geq 1 + P_1$

Result for the 3 × 3 many-to-one IC



$P_1 = P_2 = P_3 = 3\text{dB}$

Summary

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Many-to-one XC

- Strategies where a subset of interfering signals are treated as noise
- Conditions for sum-rate optimality
- 3×3 case
- $K \times K$ case

Many-to-one IC

- Strategies MIk and conditions for sum-rate optimality

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Possible Extensions

- More general topologies
  - Bounds using many-to-one results
  - Approximate sum-rate optimality
  - Recent results for strategy M1 (TIN) by Geng, Sun & Jafar 2014

Thank you

http://www.ee.iitm.ac.in/~skrishna/