E4215: Analog Filter Synthesis and Design
Frequency Transformation

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$$
\begin{aligned}
S &= \Sigma + j\Omega && \text{Prototype frequency variable} \\
s &= \sigma + j\omega && \text{Transformed frequency variable} \\
H_0(S) &= \frac{\prod_{k=1}^{m}\left(1 - \frac{S}{Z_k}\right)}{\prod_{k=1}^{n}\left(1 - \frac{S}{P_k}\right)} && \text{Prototype transfer function} \\
H(s) &= \frac{\prod_{k=1}^{M}\left(1 - \frac{s}{z_k}\right)}{\prod_{k=1}^{N}\left(1 - \frac{s}{p_k}\right)} && \text{Transformed transfer function}
\end{aligned}
$$

The prototype transfer function $H_0(s)$ has n poles, m finite zeros, and $n-m$ zeros at infinity[1]. A_p and A_s are the attenuation of the prototype filter at Ω_p and Ω_s. $A_p = -20\log_{10}|H_0(\Omega_p)|$, $A_s = -20\log_{10}|H_0(\Omega_s)|$.

1 Lowpass to Lowpass transformation

- Transformation

$$
\begin{aligned}
\frac{S}{\Omega_p} &\leftrightarrow \frac{s}{\omega_p} \\
S = j\Omega &\leftrightarrow s = j\omega = j\frac{\Omega\omega_p}{\Omega_p}
\end{aligned}
$$

- Scaled poles and zeros

$$
\begin{aligned}
Z_k &\leftrightarrow \frac{Z_k\omega_p}{\Omega_p} \\
P_k &\leftrightarrow \frac{P_k\omega_p}{\Omega_p}
\end{aligned}
$$

- Every real pole P results in a scaled real pole p.
- Every complex conjugate pole pair $P_r \pm jP_i$ results in a complex conjugate pole pair $p_r \pm jp_i$. The prototype and the transformed pole pairs have the same quality factor.
- Resulting filter has $N = n$ poles and $M = m$ finite zeros.

[1] Usually not mentioned explicitly

2 Lowpass to Highpass transformation

- Transformation

$$\frac{S}{\Omega_p} \leftrightarrow \frac{\omega_p}{s}$$
$$S = j\Omega \leftrightarrow s = -j\frac{\Omega_p\omega_p}{\Omega}$$

- "Inverted" poles and zeros

$$Z_k \leftrightarrow z_k = \frac{\Omega_p\omega_p}{Z_k}$$
$$P_k \leftrightarrow p_k = \frac{\Omega_p\omega_p}{P_k}$$

- Every real pole P results in a scaled real pole p.
- Every complex conjugate pole pair $P_r \pm jP_i$ results in a complex conjugate pole pair $p_r \pm jp_i$. The prototype and the transformed pole pairs have the same quality factor.
- Resulting filter has $N = n$ poles. The $n - m$ zeros at infinity move to the origin.

3 Lowpass to Bandpass transformation

- Transformation: $\Omega = 0$ (dc) transforms to $\omega = \omega_0$ (geometric center of the passband). Every frequency is transformed into two frequencies whose geometric mean is ω_0. i.e. if there is a peak at Ω_{peak} in the prototype response, the transformed response has two peaks at ω_{p1} and ω_{p2} where $\omega_{p1}\omega_{p2} = \omega_0^2$.

$$\frac{S}{\Omega_p} \leftrightarrow \frac{1}{\omega_b}\frac{s^2 + \omega_0^2}{s}$$
$$j\Omega \leftrightarrow j\frac{\Omega_p}{\omega_b}\frac{\omega^2 - \omega_0^2}{\omega}$$

- ω_b is the bandwidth, the width of the passband $|\omega_{p1} - \omega_{p2}|$, within which the attenuation is less than A_p.
- Every real pole P is transformed into a complex conjugate pole pair $p_r \pm jp_i$.
- Every complex conjugate pole pair $P_r \pm jP_i$ is transformed into two complex conjugate pole pairs $p_{r1} \pm jp_{i1}$ and $p_{r2} \pm jp_{i2}$ both of which have the same quality factor Q. The quality factor of the transformed pole pair increases as the ratio ω_0/ω_b increases.
- Resulting filter has $N = 2n$ poles. The order is doubled.

4 Lowpass to Band elimination transformation

- Transformation: $\Omega = 0$ (dc) transforms to $\omega = \omega_0$ (geometric center of the stopband). Every frequency is transformed into two frequencies whose geometric mean is ω_0. i.e. if there is a peak at Ω_{peak} in the prototype response, the transformed response has two peaks at ω_{p1} and ω_{p2} where $\omega_{p1}\omega_{p2} = \omega_0^2$.

$$\begin{aligned} \frac{S}{\Omega_p} &\leftrightarrow \omega_b \frac{s}{s^2 + \omega_0^2} \\ j\Omega &\leftrightarrow -j\Omega_p\omega_b \frac{\omega}{\omega^2 - \omega_0^2} \end{aligned}$$

- ω_b is the width of the band, $|\omega_{p1} - \omega_{p2}|$, within which the attenuation is more than A_p.
- Every real pole P is transformed into a complex conjugate pole pair $p_r \pm jp_i$.
- Every complex conjugate pole pair $P_r \pm jP_i$ is transformed into two complex conjugate pole pairs $p_{r1} \pm jp_{i1}$ and $p_{r2} \pm jp_{i2}$ both of which have the same quality factor Q. The quality factor of the transformed pole pair increases as the ratio ω_0/ω_b increases.
- Resulting filter has $N = 2n$ poles. The order is doubled.

5 Using frequency transformation to synthesize filters

- If it is a bandpass or a band elimination filter, convert the specified frequencies $\omega'_{p1,p2}$, $\omega'_{s1,s2}$ to $\omega_{p1,p2}$, $\omega_{s1,s2}$ which have the same geometric mean ω_0 ($\omega_{p1}\omega_{p2} = \omega_{s1}\omega_{s2} = \omega_0^2$). While doing so, the specifications should be tightened[2], not loosened.
- Translate the given specifications A_s, A_p, ω_p, ω_s (or $\omega_{p1,p2}$, $\omega_{s1,s2}$) to a lowpass prototype specification A_s, A_p, Ω_p, Ω_s. The choice of Ω_p or Ω_s depends on the available filter tables.
- Look up[3] the filter transfer function that satisfies A_s, A_p, Ω_p, Ω_s. There are usually several types of filters. The choice depends on complexity of the active realization or additional specs., e.g. group delay, if present.
- If a cascade structure is being designed, factorize the transfer functions into first and second order terms in the numerator and the denominator. Transform the prototype transfer function into the desired transfer function. Realize each pole/pole-pair (with associated zeros) using opamp-RC or g_m-C first and second order structures.
- If a ladder structure is being designed, look up the corresponding prototype ladder structure. Transform the passive structure (Fig. 1) into the desired filter. Realize the resulting structure using element replacement or leapfrog synthesis.
- Simulate the resulting active structure with ideal components to verify the integrity of the design. Resimulate with nonidealities and modify/improve the circuit if need be.

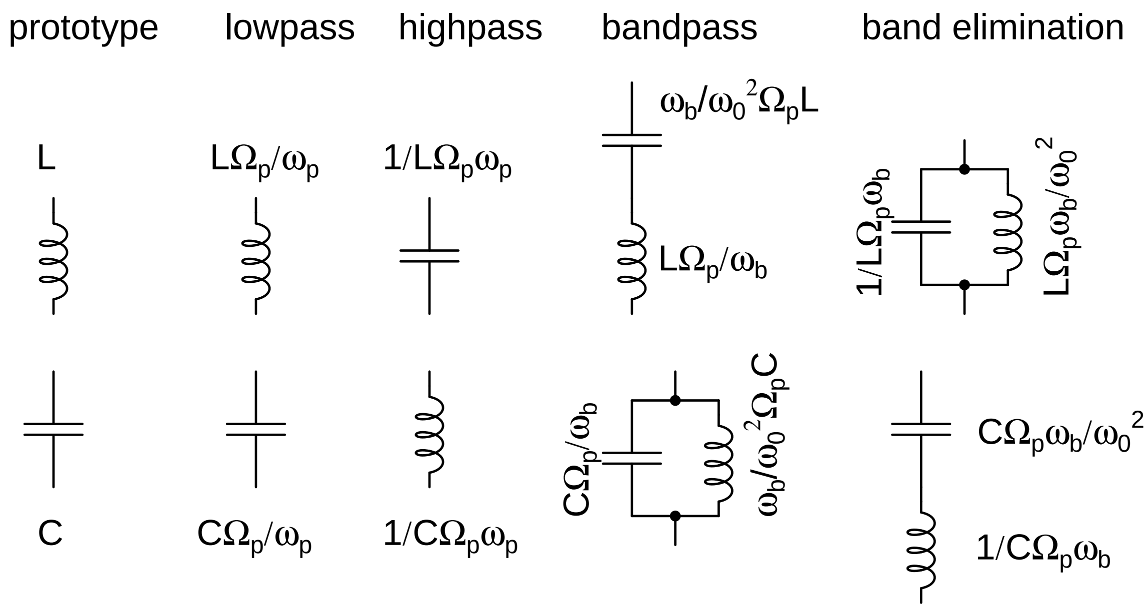


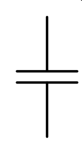
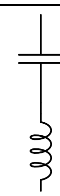
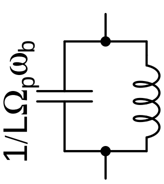
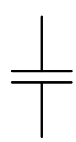
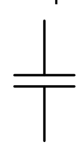
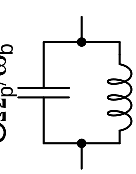
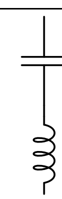
Figure 1: Transformation of passive elements

A filter design software package can eliminate one or more steps in the synthesis. You could get the poles and zeros and the passive ladder structure directly from the specifications. Mathematical tools like MATLAB can provide you poles and zeros for a variety of standard filter types. Note that unnormalized coefficients, as provided by MATLAB, can have a *very* wide range[4], especially in high order filters and can lead to gross errors in the frequency response. In those cases, the tool can be used to design a low frequency prototype and the resulting poles and zeros scaled up.

[2] The transition bandwidth will decrease on one of the sides

[3] A. I. Zverev, *Handbook of Filter Synthesis*, Wiley, New York, 1967. The mother of all filter tables!

[4] e.g. The denominator of a second order filter with $\omega_p = 1$ Grad/s and $Q = 1$ is $10^{-18}s + 10^{-9}s + 1$.

	Low pass prototype	Low pass	High pass	Band pass	Band stop
passband attenuation	$\leq A_p$ dB				
stopband attenuation	$\geq A_s$ dB				
passband edge(s)	Ω_p	ω_p	ω_p	ω_{p1}, ω_{p2}	ω_{p1}, ω_{p2}
stopband edge(s)	Ω_s	ω_s	ω_s	ω_{s1}, ω_{s2}	ω_{s1}, ω_{s2}
frequency variable	$S = \Sigma + j\Omega$	$s = \sigma + j\omega$			
Equivalence	$\frac{S}{\Omega_p}$	$\frac{s}{\omega_p}$	$\frac{\omega_p}{s}$	$\frac{1}{\omega_b}\frac{s^2+\omega_0^2}{s}$	$\omega_b\frac{s}{s^2+\omega_0^2}$
Equivalence	$\frac{\Omega}{\Omega_p}$	$\frac{\omega}{\omega_p}$	$-\frac{\omega_p}{\omega}$	$\frac{1}{\omega_b}\frac{\omega^2-\omega_0^2}{\omega}$	$-\omega_b\frac{\omega}{\omega^2-\omega_0^2}$
parameters				$\omega_0 = \sqrt{\omega_{p1}\omega_{p2}} = \sqrt{\omega_{s1}\omega_{s2}}$ $\omega_b = \omega_{p2} - \omega_{p1}$	
passband "center"	$\Omega = 0$	$\omega = 0$	$\omega = \infty$	$\omega = \pm\omega_0$	$\omega = 0, \infty$
stopband "center"	$\Omega = \infty$	$\omega = \infty$	$\omega = 0$	$\omega = 0, \infty$	$\omega = \pm\omega_0$
passband/stopband	$\frac{\Omega_p}{\Omega_s}$	$\frac{\omega_p}{\omega_s}$	$\frac{\omega_s}{\omega_p}$	$\frac{\omega_{p2}-\omega_{p1}}{\omega_{s2}-\omega_{s1}}$	$\frac{\omega_{s2}-\omega_{s1}}{\omega_{p2}-\omega_{p1}}$
inductor	L 	$L\Omega_p/\omega_p$ 	$1/L\Omega_p\omega_p$ 	 $\omega_b/\omega_0^2\Omega_p L$ $L\Omega_p/\omega_b$	 $1/L\Omega_p\omega_b$ $L\Omega_p\omega_b/\omega_0^2$
capacitor	C 	$C\Omega_p/\omega_p$ 	$1/C\Omega_p\omega_p$ 	 $C\Omega_p/\omega_b$ $\omega_b/\omega_0^2\Omega_p C$	 $C\Omega_p\omega_b/\omega_0^2$ $1/C\Omega_p\omega_b$