EE5110: Probability Foundations for Electrical Engineers **July-November 2015**

Lecture 8: The Infinite Coin Toss Model

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In this lecture, we will discuss the random experiment where each trial consists of tossing a coin infinite times. We will describe the sample space, an appropriate σ-algebra, and a probability measure that intuitively corresponds to fair coin tosses. If we denote Heads/Tails with 0/1, the sample space of this experiment turns out to be $\Omega = \{0,1\}^\infty$, and each elementary outcome is some infinite binary string. As we have seen before, this is an uncountable sample space, so defining a useful σ-algebra on Ω takes some effort.

8.1 A σ-algebra on $\Omega = \{0,1\}^\infty$

Let $\mathcal{F}_n$ be the collection of subsets of Ω whose occurrences can be decided by looking at the result of the first n tosses. More formally, the elements of $\mathcal{F}_n$ can be described as follows: $A \in \mathcal{F}_n$ if and only if there exists some $A^{(n)} \subseteq \{0,1\}^n$ such that $A = \{\omega \in \Omega | (\omega_1, \omega_2, ..., \omega_n) \in A^{(n)}\}$.

Examples:

1 Let A_1 be the set of all elements of Ω such that there are exactly 2 heads during the first 4 coin tosses. Clearly, $A_1 \in \mathcal{F}_4$.

2 Let A_2 be the set of all elements of Ω such that the third toss is a Head. Then, $A_2 \in \mathcal{F}_3$.

Also note that the following relation holds:

$$\mathcal{F}_n \subseteq \mathcal{F}_{n+1} \;\; \forall n \in \mathbb{N}. \tag{8.1}$$

Although $\mathcal{F}_n$ is a σ-algebra, it has the drawback that it allows us to describe only those subsets which can be decided in n tosses. For example, the singleton set containing all Heads is not an element of $\mathcal{F}_n$ for any n.

In order to overcome this drawback, we define $\mathcal{F}_0 = \bigcup_{i \in \mathbb{N}} \mathcal{F}_i$. In words, $\mathcal{F}_0$ is the collection of all subsets of Ω that can be decided in *finitely many* coin tosses, since an element of $\mathcal{F}_0$ must be an element of $\mathcal{F}_i$ for some $i \in \mathbb{N}$.

Proposition 8.1 *We claim the following:*

(i) $\mathcal{F}_0$ *is an algebra.*

(ii) $\mathcal{F}_0$ *is not a σ-algebra.*

Proof:

(i) This is just definition chasing! (*Left as an exercise*).

(ii) Consider the following example: Let $E = \{\omega \in \Omega \mid \text{every odd toss results in Heads}\}$. Clearly, $E \notin \mathcal{F}_0$ since we cannot decide the occurrence of E in finitely many tosses. On the other hand, E can be expressed as a countable intersection of elements in $\mathcal{F}_0$:

$$E = \bigcap_{i=1}^{\infty} A_{2i-1},$$

where $A_i \in \mathcal{F}_0$ is the set of all binary strings with Heads in the ith toss.

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Next, consider the smallest σ-algebra containing all the elements of $\mathcal{F}_0$, i.e., define

$$\mathcal{F} = \sigma(\mathcal{F}_0).$$

8.2 A Probability Measure on $(\Omega = \{0,1\}^\infty, \mathcal{F})$

Now, we shall define a uniform probability measure on $\mathcal{F}$ that corresponds to a 'fair' coin toss model. We shall first define a finitely additive function $\mathbb{P}_0$ on $\mathcal{F}_0$ that also satisfies $\mathbb{P}_0(\Omega) = 1$. Then, we shall subsequently extend $\mathbb{P}_0$ to a probability measure $\mathbb{P}$ on $\mathcal{F}$.

If $A \in \mathcal{F}_0$, then by the definition of $\mathcal{F}_0$, $\exists n$ such that $A \in \mathcal{F}_n$. By the definition of $\mathcal{F}_n$, we know that for every $A \in \mathcal{F}_n$, there exists a corresponding $A^{(n)} \subseteq \{0,1\}^n$. We will use this $A^{(n)}$ in the definition of $\mathbb{P}_0$. We define $\mathbb{P}_0 : \mathcal{F}_0 \to [0,1]$ as follows:

$$\mathbb{P}_0(A) = \tfrac{|A^{(n)}|}{2^n}.$$

Having defined $\mathbb{P}_0$ this way, we need to verify that this definition is consistent. In particular, we note that if $A \in \mathcal{F}_n$, $A \in \mathcal{F}_{n+1}$, which is trivially true because $\mathcal{F}_n's$ are nested increasing. We therefore need to prove that when we apply the definition $\mathbb{P}_0(A)$ for different choices of n, we obtain the same value. We leave it to the reader to supply a formal proof for the consistency of $\mathbb{P}_0$. However, we illustrate this consistency using the examples provided in Section 8.1.

(i) The occurrence of the event A_2 can be decided in the first 3 tosses. So, $A_2 \in \mathcal{F}_3$. The elements in $A^{(3)} \subseteq \{0,1\}^3$ corresponding to the event A_2 are $\{(0,0,0),(0,1,0),(1,0,0),(1,1,0)\}$. So $|A^{(3)}| = 4$. So, $\mathbb{P}_0(A_2) = \frac{4}{2^3} = \frac{1}{2}$.
The event A_2 can also be looked as an event in $\mathcal{F}_4$ since, $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$. The elements in the corresponding $A^{(4)}$ will be $\{(0,0,0,0),(0,1,0,0),(1,0,0,0),(1,1,0,0),(0,0,0,1),(0,1,0,1),(1,0,0,1),(1,1,0,1)\}$. So $|A^{(4)}| = 8$. So, $\mathbb{P}_0(A_2) = \frac{8}{2^4} = \frac{1}{2}$.

(ii) A_1 can be decided by looking at the outcome of the first four tosses. So, $A_1 \in \mathcal{F}_4$. It is easy to see that the number of elements in $A^{(4)} \subseteq \{0,1\}^4$ corresponding to the event A_1 that has exactly two heads is $\binom{4}{2}$. Hence, $\mathbb{P}_0(A_1) = \frac{\binom{4}{2}}{2^4}$. Next, can you compute $\mathbb{P}_0(A_1)$ by considering A_1 as an element of, say $\mathcal{F}_5$?

From the above examples, we can observe that

(a) The definition of $\mathbb{P}_0$ is consistent over different choices on n namely $n = 3$ and $n = 4$ for a given set A_2.

(b) The definition of $\mathbb{P}_0$ is also consistent with the intuition of a fair coin toss model with probability of heads being $\frac{1}{2}$.

It can be easily verified that $\mathbb{P}_0(\Omega) = 1$ and $\mathbb{P}_0$ is finitely additive. It also turns out that $\mathbb{P}_0$ is countably additive on $\mathcal{F}_0$ (the proof of this fact is non-trivial and is omitted here). This allows us to invoke the Caratheodory extension theorem and extend $\mathbb{P}_0$ to $\mathbb{P}$, a legitimate probability measure on $(\Omega, \mathcal{F})$ which agrees with $\mathbb{P}_0$ on $\mathcal{F}_0$. In other words, there exists a unique probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$.

As an example, let us consider the event E that is defined above (i.e. the set of strings in which all the odd tosses are heads). As $E \notin \mathcal{F}_0$, $\mathbb{P}_0$ is not defined for the event E. However, it is clear that $E \in \mathcal{F}$, so that $\mathbb{P}$ is defined for E. Let us calculate the probability of the event E. Recall that

$$E = \bigcap_{i=1}^{\infty} A_{2i-1}, \text{ where } A_i \quad = \quad \{\underline{\omega} \in \Omega \mid \omega_i = 0\}.$$

Let us define the event $E_m = \bigcap_{i=1}^{m} A_{2i-1}$. In other words, E_m is set of outcomes in which the first $2m$ tosses have the property of all odd tosses being heads. We can easily verify that $\mathbb{P}(E_m) = \mathbb{P}_0(E_m) = \frac{1}{2^m}$. Note that $\{E_m, m \geq 1\}$ is a sequence of nested decreasing events i.e., $E_m \supseteq E_{m+1},\ \forall m \geq 1$. It can be easily verified that E can be expressed in terms of these decreasing nested events as $E = \bigcap_{m=1}^{\infty} E_m$.

Thus,

$$\begin{aligned}
\mathbb{P}(E) &= \mathbb{P}\left(\bigcap_{m=1}^{\infty} E_m\right) \\
&\overset{(a)}{=} \lim_{m\to\infty} \mathbb{P}\left(E_m\right) \\
&= \lim_{m\to\infty} \frac{1}{2^m} \\
&= 0,
\end{aligned}$$

where the equality (a) follows from the continuity of probability measures.

8.3 Exercises

1. Show that $\mathcal{F}_n$ (defined in equation 8.1) is a σ-algebra $\forall n \in \mathbb{N}$.

2. Recall the infinite coin toss model with $\Omega = \{0,1\}^{\infty}$; where '0' denotes heads and '1' denotes tails. Define $\mathcal{F}_n$ as the collection of subsets of Ω whose occurrence can be decided by looking at the results of the first n tosses. *Exercise:*

(a) Show that $\mathcal{F}_n$ is a σ-algebra.

It turns out that the σ-algebra $\mathcal{F}_n$ for any fixed n is too small; after all, it can only serve to model the first n tosses. Let us define

$$\mathcal{F}_0 = \bigcup_{i=1}^{\infty} \mathcal{F}_n. \tag{8.2}$$

(b) Give a verbal description of the collection $\mathcal{F}_0$.

(c) Show that $\mathcal{F}_0$ is an algebra on Ω.

(d) Consider the subset $\underline{A} \subset \Omega$ consisting of sequences in which Tails occurs infinitely many times. Does $\underline{A} \in \mathcal{F}_0$?

(e) Is A^c countable?

(f) Let B be the set of all infinite sequences for which $\omega_n = 0$ for every odd n; i.e., every odd numbered toss is Heads. Show that B can be written as a countable intersection of subsets in $\mathcal{F}_0$, but $B \notin \mathcal{F}_0$. Therefore $\mathcal{F}_0$ is not a σ-algebra.

Define $\mathcal{F} = \sigma(\mathcal{F}_0)$, the σ-algebra generated by $\mathcal{F}_0$.

(g) Show that every singleton $\{\omega\}$ is $\mathcal{F}$ measurable. Show that the uniform measure on $(\Omega, \mathcal{F})$ defined in class assigns zero probability measure to singletons.

(h) Let A_i be the set of all outcomes such that the i^{th} toss is Tails. Note that $A_i \in \mathcal{F}_0$. Show that $\underline{A}$ in part (e) can be written as

$$\underline{A} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i \tag{8.3}$$

Hence show that $\underline{A}$ is $\mathcal{F}$ measurable.What is $\mathbb{P}\left\{\underline{A}\right\}$ under the uniform measure?

(i) Let $T \subset \Omega$ be the set of all coin toss sequences in which the fraction of Tails is exactly $1/2$: More precisely,

$$T = \{\omega \in \Omega \ : \ \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \omega_i}{n} = \frac{1}{2}\} \tag{8.4}$$

The set T is called the strong-law truth set, for reasons that will become clear later. Does $T \in \mathcal{F}_0$?

(j) Show that T can be expressed as

$$T = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{\omega \in \Omega \mid \frac{\sum_{i=1}^{n} \omega_i}{n} - \frac{1}{2} \mid < \frac{1}{k}\} \tag{8.5}$$

Argue that the subset inside the nested union and intersection above belongs to $\mathcal{F}_0$: Hence show that T is $\mathcal{F}$-measurable. Hint: Don't get intimidated by the multiple unions and intersections! Write-out the limit in the definition of T as the set of all $\omega \in \Omega$ such that for all $k \geq 1$; there exists an m for which for all $n > m$; we have

$$\mid \frac{\sum_{i=1}^{n} \omega_i}{n} - \frac{1}{2} \mid < \frac{1}{k} \tag{8.6}$$