EE5110: Probability Foundations in Signal Processing **July-November 2015**

Lecture 6: Discrete Probability Spaces

Lecturer: Dr. Krishna Jagannathan *Scribe: Ravi Kolla*

6.1 Discrete Probability Spaces

In this lecture, we discuss discrete probability spaces. This corresponds to the case when the sample space Ω is countable. This is the most conceptually straightforward case, since it is possible to assign probabilities to *all* subsets of Ω.

Definition 6.1 *A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a discrete probability space if the following conditions hold:*

(a) The sample space Ω is finite or countably infinite,

(b) The σ-algebra is the set of all subsets of Ω, i.e., $\mathcal{F} = 2^{\Omega}$, and

(c) The probability measure, $\mathbb{P}$, is defined for every subset of Ω. In particular, it can be defined in terms of the probabilities $\mathbb{P}(\{\omega\})$ of the singletons corresponding to each of the elementary outcomes ω, and satisfies for every $A \in \mathcal{F}$,

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\{\omega\}),$$

and

$$\sum_{\omega \in \Omega} \mathbb{P}(\{\omega\}) = 1.$$

6.1.1 Examples of Discrete Probability Space

1. Let us consider a coin toss experiment with the probability of getting a head as p and the probability of getting a tail as $(1-p)$. Then, the sample space and the σ-algebra are

$$\Omega = \{H, T\} \equiv \{0, 1\}, \qquad \mathcal{F} = 2^{\Omega} = \{\Phi, \{H\}, \{T\}, \{\Omega\}\}.$$

respectively. The probability measure is

$$\mathbb{P}(\{H\}) \equiv \mathbb{P}(\{0\}) = p,$$
$$\mathbb{P}(\{T\}) \equiv \mathbb{P}(\{1\}) = 1 - p.$$

In this case, we say that $\mathbb{P}(.)$ is a Bernoulli measure on $\left(\{0,1\}, 2^{\{0,1\}}\right)$.

2. Let $\Omega = \mathbb{N}$, $\mathcal{F} = 2^{\mathbb{N}}$. Then, we can define the probability of a singleton as

$$\mathbb{P}(\{k\}) = a_k \geq 0, k \in \mathbb{N}$$

under the constraint that

$$\sum_{k \in \mathbb{N}} \mathbb{P}(\{k\}) = 1.$$

For example, $a_k = \frac{1}{2^k}, \;\; k \in \mathbb{N}$ is a valid measure, since

$$\sum_{k\in\mathbb{N}} \frac{1}{2^k} = 1.$$

As another example, consider $a_k = (1-p)^{k-1}p, \;\; 0 < p < 1, \;\; k \in \mathbb{N}$. This is known as a geometric measure with parameter p. It is a valid probability measure since

$$\sum_{k\in\mathbb{N}} (1-p)^{k-1} p = 1.$$

3. Let $\Omega = \mathbb{N} \cup \{0\}$, $\mathcal{F} = 2^{\Omega}$. Let us define

$$\mathbb{P}(\{k\}) = \frac{e^{-\lambda}\lambda^k}{k!}, \;\; \lambda > 0.$$

   This probability measure is called a Poisson measure with parameter λ on $\left(\Omega, 2^{\Omega}\right)$. This is a valid probability measure, since

$$\sum_{k=0}^{\infty} \mathbb{P}(\{k\}) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{e^{\lambda}} = 1.$$

4. Let $\Omega = \{0, 1, 2, \cdots, N\}, \;\; N \in \mathbb{N} \;\; \mathcal{F} = 2^{\Omega}$. Let us define

$$\mathbb{P}(\{k\}) = \binom{N}{k} p^k (1-p)^{N-k}, \;\; 0 < p < 1.$$

   This probability measure is called a Binomial measure with parameters (N, p) on $\left(\Omega, 2^{\Omega}\right)$. This can be verified to be a valid probability measure as follows:

$$\sum_{k\in\Omega} \binom{N}{k} p^k (1-p)^{N-k} = (p + 1 - p)^N = 1$$

Note that in all the examples above, we have not explicitly specified an expression for $\mathbb{P}(A)$ for every $A \subset \Omega$. Since the sample space is countable, the probability of any subset of the sample space can be obtained as the sum of probabilities of the corresponding elementary outcomes. In other words, for discrete probability spaces, it suffices to specify the probabilities of singletons corresponding to each of the elementary outcomes.

6.2 Exercises

1. An urn contains a number of white balls and b number of black balls. Balls are drawn randomly from the urn without replacement. Find the probability that a white ball is drawn at the kth draw.

2. An urn contains white and black balls. When two balls are drawn without replacement, suppose the probability that both the balls are white is $\frac{1}{3}$.
(a) Find the smallest number of balls in the urn.
(b) How small can the total number of balls be if the number of black balls is even?

3. Consider the sample space $\Omega = \mathbb{N}$. Find the values of the constant C for which the following are probability measures:
(a) $f(x) = C2^{-x}$
(b) $f(x) = \frac{C2^{-x}}{x}$
(c) $f(x) = Cx^{-2}$
(d) $f(x) = \frac{C2^{x}}{x!}$

4. Recall the Poisson measure on $(\Omega, 2^{\Omega})$, where $\Omega = \mathbb{N} \cup \{0\}$. What is the probability assigned to the set of odd numbers? Prime numbers?