EE5110: Probability Foundations for Electrical Engineers **July-November 2015**

Lecture 5: Properties of Probability Measures

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5.1 Properties

In this lecture, we will derive some fundamental properties of probability measures, which follow directly from the axioms of probability. In what follows, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

- **Property 1:-** Suppose A be a subset of Ω such that $A \in \mathcal{F}$. Then,

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A). \tag{5.1}$$

  Proof:- Given any subset $A \in \Omega$, A and A^c partition the sample space. Hence, $A^c \cup A = \Omega$ and $A^c \cap A = \emptyset$. By the "*Countable Additivity*" axiom of probability, $\mathbb{P}(A^c \cup A) = \mathbb{P}(A) + \mathbb{P}(A^c)$ $\implies \mathbb{P}(\Omega) = \mathbb{P}(A) + \mathbb{P}(A^c) \implies \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

- **Property 2:-** Consider events A and B such that $A \subseteq B$ and $A, B \in \mathcal{F}$. Then $\mathbb{P}(A) \leq \mathbb{P}(B)$
  Proof:- The set B can be written as the union of two disjoint sets A and $A^c \cap B$. Therefore, we have $\mathbb{P}(A) + \mathbb{P}(A^c \cap B) = \mathbb{P}(B) \implies \mathbb{P}(A) \leq \mathbb{P}(B)$ since $\mathbb{P}(A^c \cap B) \geq 0$.

- **Property 3:-** (*Finite Additivity*) If A_1, A_2, ..., A_n are finite number of disjoint events, then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i). \tag{5.2}$$

  Proof:- This property follows directly from the axiom of *countable additivity* of probability measures. It is obtained by setting the events A_{n+1}, A_{n+2}, ... as empty sets. LHS will simplify as:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right).$$

  RHS can be manipulated as follows:

$$\begin{aligned}
\sum_{i=1}^{\infty} \mathbb{P}(A_i) &\overset{(a)}{=} \lim_{k\to\infty} \sum_{i=1}^{k} \mathbb{P}(A_i) \\
&= \sum_{i=1}^{n} \mathbb{P}(A_i) + \lim_{k\to\infty} \sum_{i=n+1}^{k} \mathbb{P}(A_i) \\
&\overset{(b)}{=} \sum_{i=1}^{n} \mathbb{P}(A_i) + \lim_{k\to\infty} 0 \\
&= \sum_{i=1}^{n} \mathbb{P}(A_i).
\end{aligned}$$

  where (a) follows from the definition of an infinite series and (b) is a consequence of setting the events from A_{n+1} onwards to null sets.

- **Property 4:-** For any A, $B \in \mathcal{F}$,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B). \tag{5.3}$$

In general, for a family of events $\{A_i\}_{i=1}^n \subset \mathcal{F}$,

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) + ... + (-1)^{n+1}\mathbb{P}\left(\bigcap_{i=1}^n A_i\right). \tag{5.4}$$

This property is proved using induction on n. The property can be proved in a much more simpler way using the concept of *Indicator Random Variables*, which will be discussed in the subsequent lectures.
Proof of Eq (5.3)**:-** The set $A \cup B$ can be written as $A \cup B = A \cup (A^c \cap B)$. Since A and $A^c \cap B$ are disjoint events, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(A^c \cap B)$. Now, set B can be partitioned as, $B = (A \cap B) \cup (A^c \cap B)$. Hence, $\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B)$. On substituting this result in the expression of $\mathbb{P}(A \cup B)$, we will obtain the final result that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

- **Property 5:-** If $\{A_i, i \geq 1\}$ are events, then

$$\mathbb{P}\left(\bigcup_{i=1}^\infty A_i\right) = \lim_{m\to\infty} \mathbb{P}\left(\bigcup_{i=1}^m A_i\right). \tag{5.5}$$

This result is known as *continuity of probability measures.*

Proof:- Define a new family of sets $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, ..., $B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$, ...
Then, the following claims are placed:
Claim 1:- $B_i \cap B_j = \emptyset$, $\forall i \neq j$.
Claim 2:- $\bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty B_i$.
Since $\{B_i, i \geq 1\}$ is a disjoint sequence of events, and using the above claims, we get

$$\mathbb{P}\left(\bigcup_{i=1}^\infty A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^\infty B_i\right) = \sum_{i=1}^\infty \mathbb{P}(B_i).$$

Therefore,

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{i=1}^\infty A_i\right) &= \sum_{i=1}^\infty \mathbb{P}(B_i) \\
&\stackrel{(a)}{=} \lim_{m\to\infty} \sum_{i=1}^m \mathbb{P}(B_i) \\
&\stackrel{(b)}{=} \lim_{m\to\infty} \mathbb{P}\left(\bigcup_{i=1}^m B_i\right) \\
&\stackrel{(c)}{=} \lim_{m\to\infty} \mathbb{P}\left(\bigcup_{i=1}^m A_i\right).
\end{aligned}$$

Here, (a) follows from the definition of an infinite series, (b) follows from *Claim 1* in conjunction with *Countable Additivity* axiom of probability measure and (c) follows from the intermediate result required to prove *Claim 2.*
Hence proved.

- **Property 6:-** If $\{A_i, i \geq 1\}$ is a sequence of increasing nested events i.e. $A_i \subseteq A_{i+1}$, $\forall i \geq 1$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{m\to\infty} \mathbb{P}\left(A_m\right). \tag{5.6}$$

- **Property 7:-** If $\{A_i, i \geq 1\}$ is a sequence of decreasing nested events i.e. $A_{i+1} \subseteq A_i \forall i \geq 1$, then

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{m\to\infty} \mathbb{P}\left(A_m\right). \tag{5.7}$$

  Properties 6 and 7 are said to be corollaries to Property 5.

- **Property 8:-** Suppose $\{A_i, i \geq 1\}$ are events, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i). \tag{5.8}$$

  This result is known as the *Union Bound*. This is bound is trivial if $\sum_{i=1}^{\infty} \mathbb{P}(A_i) \geq 1$ since the LHS of (5.8) is a probability of some event. This is a very widely used bound, and has several applications. For instance, the union bound is used in the probability of error analysis in Digital Communications for complicated modulation schemes.

  Proof:- Define a new family of sets $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, ..., $B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$, ...
  Claim 1:- $B_i \cap B_j = \emptyset$, $\forall i \neq j$.
  Claim 2:- $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$.
  Since $\{B_i, i \geq 1\}$ is a disjoint sequence of events, and using the above claims, we get

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(B_i).$$

  Also, since $B_i \subseteq A_i$ $\forall i \geq 1$, $\mathbb{P}(B_i) \leq \mathbb{P}(A_i)$ $\forall i \geq 1$ (using Property 2). Therefore, the finite sum of probabilities follow

$$\sum_{i=1}^{n} \mathbb{P}(B_i) \leq \sum_{i=1}^{n} \mathbb{P}(A_i).$$

  Eventually, in the limit, the following holds:

$$\sum_{i=1}^{\infty} \mathbb{P}(B_i) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

  Finally we arrive at the result,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

5.2 Exercises

1. a) Prove *Claim 1* and *Claim 2* stated in Property 5.

b) Prove Properties 6 and 7, which are corollaries of Property 5.

2. A standard card deck (52 cards) is distributed to two persons: 26 cards to each person. All partitions are equally likely. Find the probability that the first person receives all four aces.

3. Consider two events A and B such that $\mathbb{P}(A) > 1-\delta$ and $\mathbb{P}(B) > 1-\delta$, for some very small $\delta > 0$. Prove that $\mathbb{P}(A \cap B)$ is close to 1.

4. **[Grimmett]** Given events $A_1, A_2, ...A_n$, prove that,

$$\mathbb{P}(\cup_{1\leq r\leq n} A_r) \leq \min_{1\leq k\leq n} \Big(\sum_{1\leq r\leq n} \mathbb{P}(A_r) - \sum_{r:r\neq k} \mathbb{P}(A_r \cap A_k) \Big)$$

5. Consider a measurable space $(\Omega, \mathcal{F})$ with $\Omega = [0,1]$. A measure $\mathbb{P}$ is defined on the non-empty subsets of Ω (in $\mathcal{F}$), which are all of the form (a,b), $(a,b]$, $[a,b)$ and $[a,b]$, as the length of the interval, i.e., $\mathbb{P}((a,b)) = \mathbb{P}((a,b]) = \mathbb{P}([a,b)) = \mathbb{P}([a,b]) = b-a$.

   a) Show that $\mathbb{P}$ is not just a measure, but its a probability measure.

   b) Let $A_n = [\frac{1}{n+1}, 1]$ and $B_n = [0, \frac{1}{n+1}]$, for $n \geq 1$. Compute $\mathbb{P}(\cup_{i\in\mathbb{N}} A_i)$, $\mathbb{P}(\cap_{i\in\mathbb{N}} A_i)$, $\mathbb{P}(\cup_{i\in\mathbb{N}} B_i)$ and $\mathbb{P}(\cap_{i\in\mathbb{N}} B_i)$.

   c) Compute $\mathbb{P}(\cap_{i\in\mathbb{N}}(B_i^c \cup A_i^c))$.

   d) Let $C_m = [0, \frac{1}{m}]$ such that $\mathbb{P}(C_m) = \mathbb{P}(A_n)$. Express m in terms of n.

   e) Evaluate $\mathbb{P}(\cap_{i\in\mathbb{N}}(C_i \cap A_i))$ and $\mathbb{P}(\cup_{i\in\mathbb{N}}(C_i \cap A_i))$.

6. **[Grimmett]** You are given that at least one of the events A_n, $1 \leq n \leq N$, is certain to occur. However, certainly no more than two occur. If $\mathbb{P}(A_n) = p$ and $\mathbb{P}(A_n \cap A_m) = q$, $m \neq n$, then show that $p \geq \frac{1}{N}$ and $q \leq \frac{2}{N}$.