EE5110: Probability Foundations for Electrical Engineers **July-November 2015**

Lecture 4: Probability Spaces

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4.1 Introduction

Just as a point is not defined in elementary geometry, probability theory begins with two entities that are not defined. These undefined entities are a **Random Experiment** and its **Outcome**. These two concepts are to be understood intuitively, as suggested by their respective English meanings. We use these undefined terms to define other entities.

Definition 4.1 *The* **Sample Space** Ω *of a random experiment is the set of all possible outcomes of a random experiment.*

An outcome (or elementary outcome) of the random experiment is usually denoted by ω. Thus, when a random experiment is performed, the outcome $\omega \in \Omega$ is picked by the Goddess of Chance or Mother Nature or your favourite genie.

Note that the sample space Ω can be finite or infinite. Indeed, depending on the cardinality of Ω, it can be classified as follows:

1. Finite sample space
2. Countably infinite sample space
3. Uncountable sample space

It is imperative to note that for a given random experiment, its sample space is defined depending on what one is interested in observing as the outcome. We illustrate this using an example. Consider a person tossing a coin. This is a random experiment. Now consider the following three cases:

- Suppose one is interested in knowing whether the toss produces a head or a tail, then the sample space is given by, $\Omega = \{H, T\}$. Here, as there are only two possible outcomes, the sample space is said to be finite.

- Suppose one is interested in the number of tumbles before the coin hits the ground, then the sample space is the set of all natural numbers. In this case, the sample space is countably infinite and is given by, $\Omega = \mathbb{N}$.

- Suppose one is interested in the speed with which the coin strikes ground, then the set of positive real numbers forms the sample space. This is an example of an uncountable sample space, which is given by, $\Omega = \mathbb{R}^+$.

Thus we see that for the same experiment, Ω can be different based on what the experimenter is interested in.

Let us now have a look at one more example where the sample space can be different for the same experiment and you can get different answers based on which sample space you decide to choose.

Bertrand's Paradox: Consider a circle of radius r. What is the probability that the length of a chord chosen at random is greater than the length of the side of an equilateral triangle inscribed in the circle?

This is an interesting paradox and gives different answers based on different sample spaces. The entire description of Bertrand's Paradox can be found in [1].

Definition 4.2 *(Informal) An* **event** *is a subset of the sample space, to which probabilities will be assigned.*

An event is a subset of the sample space, but we emphasise that *not all subsets of the sample space are necessarily considered events*, for reasons that will be explained later. Until we are ready to give a more precise definition, we can consider events to be those "interesting" subsets of Ω, to which we will eventually assign probabilities. We will see later that whenever Ω is finite or countable, all subsets of the sample space can be considered as events, and be assigned probabilities. However, when Ω is uncountable, it is often not possible to assign probabilities to all subsets of Ω, for reasons that will not be clear now. The way to handle uncountable sample spaces will be discussed later.

Definition 4.3 *An event A is said to* **occur** *if the outcome ω, of the random experiment is an element of A, i.e., if $\omega \in A$.*

Let us take an example. Say the random experiment is choosing a card at random from a pack of playing cards. What is the sample space in this case? It is a 52 element set as each card is a possible outcome. As the sample space is finite, any subset of the sample space can be considered as an event. As a result there will be 2^{52} events (Power set of n elements has 2^n elements). An event can be any subset of the sample space which includes the empty set, all the singleton sets (containing one outcome) and collection of more than one outcomes. Listed below are a few events:

- The 7 of Hearts (1 element)
- A face Card (12 elements)
- A 2 and a 7 at the same time (0 element)
- An ace of any color (4 elements)
- A diamond card (13 elements)

Next, let us look at some nice properties that we would expect events to satisfy:

- Since the sample space Ω always occurs, we would like to have Ω as an event.
- If A is an event (i.e., a "nice" subset of the sample space to which we would like to assign a probability), it is reasonable to expect A^c to be an event as well.
- If A and B are two events, we are interested in the occurrence of at least one of them (A or B) as well as the occurrence of both of them (A and B). Hence, we would like to have $A \cup B$ and $A \cap B$ to be events as well.

The above three properties motivate a mathematical structure of subsets, known as an *algebra.*

4.2 Algebra, $\mathcal{F}_0$

Let Ω be the sample space and let $\mathcal{F}_0$ be a collection of subsets of Ω. Then, $\mathcal{F}_0$ is said to be an **algebra** (or a field) if

i. $\emptyset \in \mathcal{F}_0$.

ii. $A \in \mathcal{F}_0$, implies $A^c \in \mathcal{F}_0$.

iii. $A \in \mathcal{F}_0$ and $B \in \mathcal{F}_0$ implies $A \cup B \in \mathcal{F}_0$.

It can be shown that an algebra is closed under finite union and finite intersection (*see Exercise 1(a))*. However, a natural question that arises at this point is "Is the structure of an algebra enough to study events of typical interest?" Consider the following example:

Example:- Toss a coin repeatedly until the first heads shows. Here, $\Omega = \{$H, TH, TTH, ... $\}$. Let us say that we are interested in determining if the number of tosses before seeing a head is even. It is easy to see that this 'event' of interest will not be included in the algebra. This is because an algebra contains only finite unions of subsets, but the 'event' of interest entails a countably infinite union. This motivates the definition of a *σ-algebra.*

4.3 Sigma Algebra, $\mathcal{F}$

A collection $\mathcal{F}$ of subsets of Ω is called a **σ-algebra** (or σ-field) if

i. $\emptyset \in \mathcal{F}$.

ii. A $\in \mathcal{F}$, implies $A^c \in \mathcal{F}$.

iii. If A_1, A_2, A_3, ... is a countable collection of subsets in $\mathcal{F}$, then $\bigcup_{i=1}^{\infty} Ai \in \mathcal{F}$.

Note that unlike an algebra, a σ-algebra is closed under countable union and countable intersection (*see Exercise 1(b)*). Some examples of σ-algebra are:

i. $\{\emptyset, \Omega\}$

ii. $\{\emptyset,\ A,\ A^c,\ \Omega\}$

iii. Power set of Ω, denoted by 2^Ω.

The 2-tuple $(\Omega, \mathcal{F})$ is called a *measurable space*. Also, every member of the σ-algebra $\mathcal{F}$ is called an $\mathcal{F}$-measurable set in the context of measure theory. In the specific context of probability theory, $\mathcal{F}$-measurable sets are called *events*. Thus, whether or not a subset of Ω is considered an event depends on the σ-algebra that is under consideration.

4.4 Measure

We now proceed to define measures and measure spaces. We will see that a probability space is indeed a special case of a measure space.

Definition 4.4 *Let $(\Omega, \mathcal{F})$ be a measurable space. A* measure *on $(\Omega, \mathcal{F})$ is a function μ: $\mathcal{F} \longrightarrow [0,\infty]$ such that*

i. $\mu(\emptyset) = 0$.

ii. If $\{A_i, i \geq 1\}$ is a sequence of disjoint sets in $\mathcal{F}$, then the measure of the union (of countably infinite disjoint sets) is equal to the sum of measures of individual sets, i.e.,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \tag{4.1}$$

The second property stated above is known as the *countable additivity* property of measures. From the definition, it is clear that a measure can only be assigned to elements of $\mathcal{F}$. The triplet $(\Omega, \mathcal{F}, \mu)$ is called a *measure space.* μ is said to be a *finite measure* if $\mu(\Omega) < \infty$; otherwise, μ is said to be an *infinite measure.* In particular, if $\mu(\Omega) = 1$, then μ is said to be a *probability measure.* Next, we state this explicitly for pedagogical completeness.

4.5 Probability Measure

A *probability measure* is a function $\mathbb{P}$: $\mathcal{F} \longrightarrow [0,1]$ such that

i. $\mathbb{P}(\emptyset) = 0$.

ii. $\mathbb{P}(\Omega) = 1$.

iii. (*Countable additivity:*) If $\{A_i, i \geq 1\}$ is a sequence of disjoint sets in $\mathcal{F}$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space*, and the three properties, stated above, are sometimes referred to as the axioms of probability.

Note:- It is clear from the definition that probabilities are defined only to elements of $\mathcal{F}$, and not necessarily to all subsets of Ω. In other words, probability measures are assigned only to events. Even when we speak of the probability of an elementary outcome ω, it should be interpreted as the probability assigned to the singleton set $\{\omega\}$ (assuming of course, that the singleton is an event).

4.6 Exercises

1. a) Let $A_1, A_2, \ldots, A_n$ be a finite collection of subsets of Ω such that $A_i \in \mathcal{F}_0$ (an algebra), $1 \leq i \leq n$. Show that $\bigcup_{i=1}^{n} A_i \in \mathcal{F}_0$ and $\bigcap_{i=1}^{n} A_i \in \mathcal{F}_0$. Hence, infer that an algebra is closed under finite union and finite intersection.

   b) Suppose A_1, A_2, A_3, ... is a countable collection of subsets in the σ-algebra $\mathcal{F}$, then show that $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

2. [**σ-algebra : Properties and Construction**].
   (a) Show that a σ-algebra is also an algebra.
   (b) Given a sample space Ω and a σ-algebra $\mathcal{F}$ of the subsets of Ω, show that if $A, B \in \mathcal{F}$, $A \setminus B$ and $A \triangle B$, the symmetric difference of A and B are present in $\mathcal{F}$.
   (c) Consider the random experiment of throwing a die. If a statistician is interested in the occurrence of either an odd or an even outcome, construct a sample space and a σ-algebra of subsets of this sample space.
   (d) Let $A_1, A_2, ..., A_n$ be arbitrary subsets of Ω. Describe (explicitly) the smallest σ-algebra $\mathcal{F}$ containing $A_1, A_2, ..., A_n$. How many sets are there in $\mathcal{F}$? (Give an upper bound that is attainable under certain conditions). List all the sets in $\mathcal{F}$ for $n = 2$.
3. Let $\mathcal{F}$ and $\mathcal{G}$ be two σ-algebras of subsets of Ω.
   (a) Is $\mathcal{F} \cup \mathcal{G}$, the collection of subsets of Ω lying in either $\mathcal{F}$ or $\mathcal{G}$ a σ-algebra?
   (b) Show that $\mathcal{F} \cap \mathcal{G}$, the collection of subsets of Ω lying in both $\mathcal{F}$ and $\mathcal{G}$ is a σ-algebra.
   (c) Generalize (b) to arbitrary intersections as follows. Let $\mathcal{I}$ be an arbitrary index set (possibly uncountable), and let $\{\mathcal{F}_i\}_{i \in \mathcal{I}}$ be a collection of σ-algebras on Ω. Show that $\bigcap_{i \in \mathcal{I}} \mathcal{F}_i$ is also a σ-algebra.
4. Let $\mathcal{F}$ be a σ-algebra of subsets of Ω, and let $B \in \mathcal{F}$. Show that

   $$\mathcal{G} = \{A \cap B \mid A \in \mathcal{F}\}$$

   is a σ-algebra of subsets of B.
5. Let X and Y be two sets and let $f : X \to Y$ be a function. If $\mathcal{F}$ is a σ-algebra over the subsets of Y and $\mathcal{G} = \{ A \mid \exists\, B \in \mathcal{F} \text{ such that } f^{-1}(B) = A \}$, does $\mathcal{G}$ form a σ-algebra of subsets of X ?
   Note that $f^{-1}(N)$ is the notation used for the pre-image of set N under the function f for some $N \subseteq Y$. That is, $f^{-1}(N) = \{x \in X | f(x) \in N\}$ for some $N \subseteq Y$.
6. Let Ω be an arbitrary set.
   (a) Is the collection $\mathcal{F}_1$ consisting of all finite subsets of Ω an algebra?
   (b) Let $\mathcal{F}_2$ consist of all finite subsets of Ω, and all subsets of Ω having a finite complement. Is $\mathcal{F}_2$ an algebra?
   (c) Is $\mathcal{F}_2$ a σ-algebra?
   (d) Let $\mathcal{F}_3$ consist of all countable subsets of Ω, and all subsets of Ω having a countable complement. Is $\mathcal{F}_3$ a σ-algebra?

References

[1] Sheldon Ross, "A First Course in Probability," *Pearson*, 8th Edition.