**EE5110 : Probability Foundations for Electrical Engineers** **July-November 2015**

# Lecture 30: The Central Limit Theorem

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## 30.1 Central Limit Theorem

In this section, we will state and prove the central limit theorem. Let $\{X_i\}$ be a sequence of i.i.d. random variables having a finite variance. From law of large numbers we know that for large $n$, the sum $S_n$ is approximately as big as $n\mathbb{E}[X]$ , i.e.,

$$\frac{S_n}{n} \xrightarrow{i.p.} \mathbb{E}[X],$$

$$\Rightarrow \quad \frac{S_n - n\mathbb{E}[X]}{n} \xrightarrow{i.p.} 0.$$

Thus whenever the variance of $X_i$ is finite, the difference $S_n - n\mathbb{E}[X]$ grows slower as compared to $n$. The Central Limit Theorem (CLT) says that this difference scales as $\sqrt{n}$, and that the distribution of $\frac{S_n - n\mathbb{E}[X]}{\sqrt{n}}$ approaches a normal distribution as $n \to \infty$ *irrespective* of the distribution of $X_i$.

$$\frac{S_n - n\mathbb{E}[X]}{\sqrt{n}} \sim N(0, \sigma_X^2).$$

**Theorem 30.1 (Central Limit Theorem)** *Let $\{X_i\}$ be a sequence of i.i.d. random variables with mean $\mathbb{E}[X]$ and a non-zero variance $\sigma_X^2 < \infty$. Let $Z_n = \frac{S_n - n\mathbb{E}[X]}{\sigma_X \sqrt{n}}$. Then, we have $Z_n \xrightarrow{D} \mathcal{N}(0,1)$, i.e., $\lim\limits_{n\to\infty} F_{Z_n}(z) = \int\limits_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\, dx$, $\forall z \in \mathbb{R}$.*

**Proof:** Let $Y_n = \frac{X_n - \mathbb{E}[X]}{\sigma_X}$. Let $Z_n = \frac{\sum\limits_{i=1}^{n} Y_i}{\sqrt{n}}$. It is easy to see that $Y_n$ has unit variance and zero mean, *i.e.,* $\mathbb{E}[Y_n] = 0$ and $\sigma_{Y_n}^2 = 1$ .

$$\begin{aligned}
C_{Y_n}(t) &= 1 + it\mathbb{E}[Y_n] + \frac{i^2 t^2 \mathbb{E}[Y_n^2]}{2} + O(t^2), \\
C_{Y_n}(t) &= 1 + it(0) + \frac{i^2 t^2 (1)}{2} + o(t^2), \\
&= 1 - \frac{t^2}{2} + o(t^2), \\
C_{Z_n}(t) &= \left[ C_{Y_n}\left( \frac{t}{\sqrt{n}} \right) \right]^n, \\
&= \left[ 1 - \frac{t^2}{2n} + o\left( \frac{t^2}{n} \right) \right]^n \longrightarrow e^{\frac{-t^2}{2}} \ \forall t.
\end{aligned}$$

From the theorem on convergence of characteristic functions, $Z_n$ converges to a standard Gaussian in distribution.

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For example, if $X_i$'s are discrete random variables, the CDFs will be step functions. As $n \to \infty$, these step functions will gradually converge to the error function (i.e. the steps will gradually decrease to form a continuous distribution as $n \to \infty$).

It is also important to understand what this theorem does *not* say. It is not saying that the probability density function converges to $\frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$. Convergence in density function requires more stringent conditions which are stated in the Local Central Limit Theorem.

**Theorem 30.2 (Local Central Limit Theorem)** *Let $X_1, X_2, \ldots$ be i.i.d. random variables with zero mean and unit variance. Suppose further that their common characteristic function $\phi$ satisfies the following:*

$$\int_{-\infty}^{\infty} |\phi(t)|^r dt < \infty.$$

*for some integer $r \geq 1$. The density function $g_n$ of $U_n = \frac{(X_1+X_2+\ldots+X_n)}{\sqrt{n}}$ exists for $n \geq r$, and furthermore we have,*

$$g_n(x) \to \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}},$$

*as $n \longrightarrow \infty$, uniformly in $x \in \mathbb{R}$.*

**Proof:** For a proof, refer to Section 5.10 in [1].

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Let $X_1, X_2, \ldots$ be i.i.d. random variables with zero mean and unit variance. From CLT, we know that $U_n = \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}}$ is distributed as a standard Gaussian. We now look at yet another interesting result which deals with the largest value taken by $U_m$, $m \geq n$, for a large $n$.

**Theorem 30.3 (The Law of the Iterated Logarithm)** *Let $X_1, X_2, \ldots$ be i.i.d. random variables with zero mean and unit variance. Also, let $S_n = \sum_{i=1}^{n} X_i$ Then,*

$$\mathbb{P}\left( \limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \; log \; log \; n}} = 1 \right) = 1.$$

Unlike the CLT which talks about distribution of $U_n$ for a large, fixed $n$, law of iterated logarithm talks about the largest fluctuation in $U_m$, for $m \geq n$. In particular, it bounds the largest value taken by $U_m$ beyond $n$. Formally, the subset of $\Omega$ for which this holds has a probability measure 1.

# References

[1] G. G. D. Stirzaker and D. Grimmett. Probability and random processes. Oxford Science Publications, 2001.