EE5110: Probability Foundations for Electrical Engineers **July-November 2015**

Lecture 2: A crash course in Real Analysis

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This lecture is an introduction to Real Analysis. Here we introduce the important concepts and theorems from Real Analysis that will be useful in the rest of the course. Interested readers may refer the book listed in the References section to learn the proofs of the theorems.

2.1 Notations

$\in$ - belongs to.
$\exists$ - there exists.
$\forall$ - for all.
$\implies$ - implies.
$\mathbb{R}$ - set of real numbers.
$\mathbb{Q}$- set of rational numbers.
$\wedge$ - and.
$\mathbb{N}$ - set of natural numbers.
$\rightarrow$ - converges to.
iff- if and only if.
$\subseteq$ - is a subset of.
ϕ -null set.
$\cap$ - intersection.
i.e. - that is.

2.2 Field

A set X is a field if it satisfies the six properties listed below under the two abstract operations '+' and '.'.

Closure: If a and $b \in X$, then $a+b \in X$ and $a.b \in X$. Hence X is closed under addition and multiplication.

Commutativity: If a and $b \in X$, then $a+b = b+a$ and $a.b = b.a$. Hence X is commutative under addition and multiplication.

Associativity: If a, b and $c \in X$, then $(a+b)+c = a+(b+c)$ and $(a.b).c = a.(b.c)$. Hence X is associative under addition and multiplication.

Identity: If $a \in X$, then $\exists$ elements 0 and 1 in X such that a+0=a and a.1=a.

Inverse: If $a \in X$, then $\exists$ elements -a and a^{-1} in X such that a+(-a)=0 and $a.a^{-1}$=1. Multiplicative inverse does not exist if a=0.

Distributivity: If a, b and $c \in X$, then multiplication is distributive with respect to addition. $a.(b+c)=a.b+a.c$.

Note that, the elements 0 and 1 are unique. If $X \subseteq$ real numbers $\mathbb{R}$, then the elements in the field can also be compared. A field whose elements can be compared is called an ordered field. Another example for an ordered field is a set of rational numbers $\mathbb{Q}$. Henceforth we will concentrate only on the real field $\mathbb{R}$.

2.2.1 Order axioms

Law of trichotomy: If a, $b \in \mathbb{R}$, then a=b or $a > b$ or $a < b$.

Transitivity: Let a, b, $c \in \mathbb{R}$. If $a > b$ and $b > c$, then $a > c$.

Ordering and addition operator: Let a, b, $c \in \mathbb{R}$. $a > b \implies a+c > b+c$.

Ordering and product operator: Let a, b, $c \in \mathbb{R}$. $a > b \implies ac > bc$, if $c > 0$.

2.3 Boundedness

A subset S of $\mathbb{R}$ is bounded above if $\exists$ a real number M such that $x \leq M$, $\forall\, x \in S$. Here, M is called an upper bound of S. Similarly S is bounded below if $\exists$ a real number m such that $x \geq m$, $\forall\, x \in S$. Here, m is called a lower bound of S. A set is bounded if it is both bounded above and below. Any element greater than M and lesser than m are also upper and lower bounds of S respectively.

Supremum: *The supremum of S is the least upper bound of the set S.* More precisely, K is a supremum of S if

- K is an upper bound of S, *i.e.* , $x \leq K$, $\forall\, x \in S$.
- There exists no number less than K which is an upper bound of S, *i.e.* for any $\delta > 0$, $\exists\, z \in S$ such that $z > K$-δ.

Similarly one can define the infimum, as the greatest lower bound of a set. It is important to note that supremum and infimum need not be elements of the set. For instance, 1 is the supremum of the set (0,1), but is not an element of the set. Also, if the supremum is an element of the set itself, then it is the maximum of that set.

2.4 Completeness property

The completeness axiom or the least upper bound property is one of the fundamental properties of the real field $\mathbb{R}$.

Completeness Axiom: *Any non empty subset A of $\mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$.*

In other words, the Completeness Axiom guarantees that, for any nonempty set of $\mathbb{R}$ that is bounded above, a supremum exists. Although $\mathbb{R}$ and $\mathbb{Q}$ are ordered fields, we will see in the exercise below that the latter does not satisfy the completeness property. Indeed, completeness along with the ordered field property characterizes $\mathbb{R}$. Thus, $\mathbb{R}$ is also referred to as a complete ordered field.

We now list a few important theorems (without proofs), which are consequences of the completeness property.

Theorem 2.1 *If x and y are any two positive real numbers, then there exists a positive integer m such that $mx > y$. This is called as the Archimedean property of real numbers.*

Theorem 2.2 *Every open interval contains a rational number.*

Theorem 2.3 *Let $x \in \mathbb{R}$, $n \geq 2$, $n \in \mathbb{N}$, then*

- *If $x \geq 0$ and n is even then $\exists$ a unique $y \geq 0$ such that $y^n = x$.*
- *If $x \in \mathbb{R}$ and n is odd then $\exists$ a unique $y \in \mathbb{R}$ such that $y^n = x$.*

2.5 Sequences

A (real) sequence is a function from $\mathbb{N}$ to $\mathbb{R}$. A sequence $\{x_n\}$ of real numbers is said to converge to $x \in \mathbb{R}$ if for every $\epsilon > 0$, $\exists$ a natural number n_0 such that $|x_n - x| < \epsilon \; \forall \; n \geq n_0$.

Theorem 2.4 *Let $\{x_n\}$ be a monotonically increasing sequence such that $x_n \leq \alpha$ for some $\alpha \in \mathbb{R}$, and all $n \geq 1$. Then $\{x_n\}$ converges to a real number.*

In other words, the above theorem can be stated as: a monotonically non-decreasing sequence which is bounded above converges. The proof again uses the completeness property. The student is encouraged to attempt a proof of this theorem, before referring to a text.

Corollary 2.5 *A convergent sequence is bounded.*

Of course, a bounded sequence need not converge: consider for example, the sequence $x_n = \{(-1)^n\}$. The sequence is bounded, but does not converge. Next, we list some elementary properties of limits. Let $\{x_n\}$ and $\{y_n\}$ be two sequences that converge to x and y, respectively.

- $x_n + y_n \to x + y$.
- $\alpha x_n \to \alpha x, \; \forall \; \alpha \in \mathbb{R}$.
- $x_n y_n \to xy$.
- $x_n \geq 0 \; \forall \; n \implies x \geq 0$.
- $x_n \leq y_n \; \forall \; n \implies x \leq y$.
- If $y \neq 0$, $\frac{x_n}{y_n} \to \frac{x}{y}$.

Theorem 2.6 Sandwich/ Two policeman theorem:
If $x_n \leq z_n \leq y_n \; \forall \; n$ and if x_n and y_n converge to x, then z_n also converges to x (Prove it!).

Examples of some important sequences are as follows:

Cauchy Sequences: A sequence $\{x_n\}$ is called a Cauchy sequence if $\forall \; \epsilon > 0$, $\exists \; n_0 \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon \; \forall \; n, m \geq n_0$.

Subsequences: A subsequence of a sequence is an infinite ordered subset of that sequence. Here are a few basic theorems about subsequences.

Theorem 2.7 *Every real sequence has a monotonic subsequence.*

Theorem 2.8 Bolzano-Weistrass Theorem: *Every bounded sequence has a convergent subsequence.*

Theorem 2.9 *A sequence $\{x_n\}$ is convergent iff $\{x_n\}$ is bounded and every convergent subsequence of $\{x_n\}$ converges to the same limit.*

Theorem 2.10 *A real sequence is convergent iff it is a Cauchy sequence.*

2.6 Metric Spaces

A set X is a metric space if we can associate a real number $d(a, b)$ with any two elements a and b of the set X such that

- $d(a, b) > 0$ if $a \neq b$; $d(a, a)$=0.
- $d(a, b)$=$d(b, a)$.
- Triangle inequality: $d(a, b) \leq d(a, c) + d(b, c)$ for any $c \in X$.

Any function d that satisfies these properties on a set is called a metric.

2.6.1 Open set

Let (X,d) be a metric space. The *open ball* $B(x, r)$ centred at x of radius r is defined as $B(x, r) = \{y \in X : d(x, y) < r\}$. A set $A \subseteq X$ is said to be *open* in X if for every $x \in A$, $\exists\ r > 0$ such that $B(x, r) \subseteq A$.

Theorem 2.11 *Let (X,d) be a metric space, then*

- *X and the null set ϕ are open in X.*
- *An arbitrary union of open sets is open.*
- *A finite intersection of open sets is open.*

Definition 2.12 *Interior point: Let (X,d) be a metric space and $A \subseteq X$. A point $x \in X$ is called an interior point of A if there exists $r > 0$, such that $B(x, r) \subseteq A$.*

Let A^0 denote the set of all interior points of A. Clearly, $A^0 \subseteq A$.

Lemma 2.13 *Let (X,d) be a metric space, then*

- *A^0 is open in X.*
- *A^0 is the largest open set contained in A.*
- *A^0=A iff A is open.*

2.6.2 Closed set

Let (X,d) be a metric space and $A \subseteq X$. A is said to be *closed* in X if A^c is open in X.

Theorem 2.14 *Let (X,d) be a metric space, then*

- *X and the null set ϕ are closed in X.*
- *An arbitrary intersection of closed sets is closed.*
- *A finite union of closed sets is closed.*

Definition 2.15 *Limit point: Let (X,d) be a metric space and $A \subseteq X$. A point $x \in X$ is called a limit point of A, if for every $r > 0$, $B(x,r)$ contains at least one point of A.*

The *closure* of A, denoted $\overline{A}$, is defined as the set of all limit points of A. Clearly, $A \subseteq \overline{A}$.

Lemma 2.16 *Let (X,d) be a metric space, then*

- *$\overline{A}$ is closed in X.*
- *$\overline{A}$ is the smallest closed set containing A.*
- *$\overline{A}=A$ iff A is closed.*

2.6.3 Compact set

A subset A of a metric space X is compact if every sequence in A has a convergent subsequence in A.

Theorem 2.17 Heine-Borel Theorem:
In any Euclidean space $\mathbb{R}^d$, a set A is compact iff it is closed and bounded.

2.7 Functions

A function f maps every element in set A to a unique element in set B. Let (X, d_X) and (Y, d_Y) be two metric spaces. Let A be a subset of X and $a \in A$, and let f be a function from A to Y. The function f is said to be continuous at a if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), f(a)) < \epsilon$ for all points $x \in A$ for which $d_X(x,a) < \delta$. If f is continuous at every point on X, then f is continuous on X.

Theorem 2.18 *f is continuous iff $f(x_n)$ converges to $f(x)$ in Y whenever the sequence x_n converges to x in X.*

Theorem 2.19 *A function f that maps a metric space X into a metric space Y is continuous on X iff $f^{-1}(B)$ is open in X for every open set B in Y. ($f^{-1}(B)$ is the inverse image of set B. f^{-1} does not mean inverse function here.)*

Theorem 2.20 *A function f that maps a metric space X into a metric space Y is continuous on X iff $f^{-1}(B)$ is closed in X for every closed set B in Y.*

Theorem 2.21 *If function f is a continuous mapping of a compact metric space X into a metric space Y, then $f(X)$ is compact.*

2.8 Exercises

1. Prove the uniqueness of the supremum and infimum of a set.

2. Let $S = \{x : x \in \mathbb{Q}, x > 0 \wedge x^2 < 2\}$ be a subset of $\mathbb{Q}$. Show that S has no rational supremum. This shows that the completeness axiom does not hold for $\mathbb{Q}$.

3. Let f and g be continuous functions on metric space X, then $f + g$ and fg are continuous on X.

References

[WR] Walter Rudin, "Principles of Mathematical Analysis," *McGraw Hill International Series*, Third Edition.