EE5110: Probability Foundations for Electrical Engineers	**July-November 2015**

Lecture 24: Probability Generating Functions

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24.1 Probability Generating Functions (PGF)

Definition 24.1 *Let X be an integer valued random variable. The probability generating function (PGF) of X is defined as :*

$$G_X(z) \triangleq \mathbb{E}[z^X] = \sum_i z^i \mathbb{P}(X = i).$$

24.1.1 Convergence

For a non-negative valued random variable, there exists R, possibly $+\infty$, such that the PGF converges for $|z| < R$ and diverges for $|z| > R$ where $z \in \mathbb{C}$. $G_X(z)$ certainly converges for $|z| < 1$ and possibly in a larger region as well. Note that,

$$|G_X(z)| = \left| \sum_i z^i \mathbb{P}(X = i) \right| \leq \sum_i |z|^i.$$

This implies that $G_X(z)$ converges absolutely in the region $|z| < 1$. Generating functions can be defined for random variables taking negative as well as positive integer values. Such generating functions generally converge for values of z satisfying $\alpha < |z| < \beta$ for some α, β such that $\alpha \leq 1 \leq \beta$.

Example 1 : Consider the Poisson random variable X with probability mass function

$$\mathbb{P}(X = i) = \frac{e^{-\lambda} \lambda^i}{i!}, \quad i \geq 0.$$

Find the PGF of X.

Solution : The PGF of X is

$$G_X(z) = \sum_{i=1}^{\infty} \frac{z^i \lambda^i e^{-\lambda}}{i!} = e^{\lambda(z-1)}, \quad \forall z \in \mathbb{C}.$$

Example 2 : Consider the geometric random variable X with probability mass function

$$\mathbb{P}(X = i) = (1-p)^{i-1} p, \quad i \geq 1.$$

Find the PGF of X.

Solution : The PGF of X is

$$\begin{aligned} G_X(z) &= \sum_{i=1}^{\infty} (1-p)^{i-1} p z^i, \\ &= \frac{pz}{1 - z(1-p)}, \quad \text{if} \quad |z| < \frac{1}{1-p}. \end{aligned}$$

24.1.2 Properties

1. $G_X(1) = 1$.

2. $\frac{\mathrm{d}G_X(z)}{\mathrm{d}z}\Big|_{z=1} = \mathbb{E}[X]$.

   Proof : From definition

$$G_X(z) = \mathbb{E}\left[z^X\right] = \sum_i z^i \mathbb{P}(X = i).$$

   Now,

$$\begin{aligned}\frac{\mathrm{d}G_X(z)}{\mathrm{d}z} &= \frac{\mathrm{d}}{\mathrm{d}z}\sum_i z^i \mathbb{P}(X = i), \\ &\overset{(a)}{=} \sum_i \frac{\mathrm{d}}{\mathrm{d}z} z^i \mathbb{P}(X = i), \\ &= \sum_i i z^{i-1} \mathbb{P}(X = i),\end{aligned}$$

   where the interchange of differentiation and summation in (a) is a consequence of absolute convergence of the series $\sum_i z^i \mathbb{P}(X = i)$. Thus,

$$\frac{\mathrm{d}G_X(z)}{\mathrm{d}z}\Big|_{z=1} = \mathbb{E}[X].$$

3. $\frac{\mathrm{d}^k G_X(z)}{\mathrm{d}z^k}\Big|_{z=1} = \mathbb{E}\left[X\left(X-1\right)\left(X-2\right)\cdots\left(X-k+1\right)\right]$.

4. If X and Y are independent and $Z = X + Y$, then $G_Z(z) = G_X(z)G_Y(z)$. The ROC for the PGF of z is the intersection of the ROCs of the PGFs of X and Y.
   Proof :

$$G_Z(z) = \mathbb{E}[z^Z] = \mathbb{E}[z^{X+Y}] = \mathbb{E}[z^X.z^Y].$$

   Since X and Y are independent, they are uncorrelated. This implies that

$$\mathbb{E}[z^X.z^Y] = \mathbb{E}[z^X]\mathbb{E}[z^Y] = G_X(z)G_Y(z).$$

   Hence proved.

5. **Random sum of discrete RVs** : Let $Y = \sum_{i=1}^{N} X_i$, where X_i's are i.i.d discrete positive integer valued random variables and N is independent of X_i's. The PGF of Y is $G_Y(z) = G_N(G_X(z))$.
   Proof :

$$G_Y(z) = \mathbb{E}[z^Y] = \mathbb{E}\left[\mathbb{E}\left[z^Y|N\right]\right] \quad \text{(By law of iterated expectation).}$$

   Now,

$$\mathbb{E}\left[z^Y|N = n\right] = \mathbb{E}\left[z^{\sum_i x_i}|N = n\right] = \mathbb{E}\left[G_X(z)^N\right].$$

   This implies that

$$G_Y(z) = G_N\left(G_X(z)\right).$$

24.2 Exercise

1. Find the PMF of a random variable X whose probability generating function is given by

$$G_X(z) = \frac{(\frac{1}{3}z+\frac{2}{3})^4}{z}$$

2. Suppose there are X_0 individuals in initial generation of a population. In the n^{th} generation, the X_n individuals independently give rise to numbers of offspring $Y_1^{(n)}, Y_2^{(n)}, ..., Y_{X_n}^{(n)}$, where $Y_1^{(n)}, Y_2^{(n)}, ..., Y_{X_n}^{(n)}$ are i.i.d. random variables. The total number of individuals produced at the $(n+1)^{st}$ generation will then be $X_{n+1} = Y_1^{(n)} + Y_2^{(n)} + ... + Y_{X_n}^{(n)}$. Then, $\{X_n\}$ is called a branching process. Let X_n be the size of the n^{th} generation of a branching process with family-size probability generating function $G(z)$, and let $X_0 = 1$. Show that the probability generating function $G_n(z)$ of X_n satisfies $G_{n+1}(z) = G(G_n(z))$ for $n \geq 0$. Also, prove that $\mathbb{E}[X_n] = \mathbb{E}[X_{n-1}]G'(1)$.