EE5110: Probability Foundations for Electrical Engineers **July-November 2015**

Lecture 21: Expectation of CRVs, Fatou's Lemma and DCT

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In the present lecture, we will cover the following three topics:

- Integration of Continous Random Variables
- Fatou's Lemma
- Dominated Convergence Theorem (DCT)

21.1 Integration of Continuous Random Variables

Theorem 21.1 *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable. Let g be a measurable function which is either non-negative or satisfies $\int |g| d\mathbb{P}_X < \infty$. Then,*

$$\mathbb{E}[g(X)] = \int g f_X d\lambda.$$

In particular, if $g(x) = x$, i.e. the identity map, we have

$$\mathbb{E}[X] = \int x f_X d\lambda.$$

Proof: Let us first consider the case of g being a simple function i.e. $g = \sum_{i=1}^{K} a_i I_{A_i}$ for some measurable disjoint subsets A_i over the real line. We then have

$$\begin{aligned}
\mathbb{E}[g(X)] &= \int g \ d\mathbb{P}_X \\
&= \sum_{i=1}^{K} a_i \mathbb{P}_X(A_i) && [g \text{ is a simple function}] \\
&= \sum_{i=1}^{K} a_i \int_{A_i} f_X d\lambda && [\text{From Radon-Nikodym Theorem}] \\
&= \sum_{i=1}^{K} \int_{A_i} a_i \ f_X d\lambda && [a_i \text{ is a constant}] \\
&= \sum_{i=1}^{K} \int_{\Omega} (a_i I_{A_i} f_X) d\lambda && [I_{A_i} \text{ is the indicator random variable of event } A_i]
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega} \sum_{i=1}^{K} (a_i I_{A_i} f_X) d\lambda \quad \text{[Interchanging finite summation and integral]} \\
&= \int_{\Omega} \left(\sum_{i=1}^{K} (a_i I_{A_i}) \right) f_X d\lambda \\
&= \int_{\Omega} (g f_X) d\lambda.
\end{aligned}
$$

Thus we have proved the above theorem for simple functions. We now assume g to be a non-negative measurable function which may not necessarily be simple.

Let g_n be an increasing sequence of non-negative simple functions that converge to g point wise. One way of coming up with such a sequence was discussed in the previous lecture. We then have,

$$
\begin{aligned}
\mathbb{E}[g(X)] &= \lim_{n\to\infty} \int g_n d\mathbb{P}_X \qquad \text{[From MCT]} \\
&= \lim_{n\to\infty} \int g_n f_X d\lambda \qquad \text{[From result for simple functions]} \\
&= \int g f_X d\lambda. \qquad \text{[From MCT, since } g_n f_X \uparrow g f_X\text{]}
\end{aligned}
$$

For arbitrary g which are absolutely integrable, a similar proof can be worked out by writing $g = g_+ - g_-$ and proceeding. ■

Example 1: Let X be an exponential random variable with parameter μ. Find $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.
Solution: Recall that for an exponential random variable with parameter μ, $f_X(x) = \mu e^{-\mu x}$. Thus, we have

$$\mathbb{E}[X] = \int x f_X d\lambda = \int_0^{\infty} x \mu e^{-\mu x} dx = \frac{1}{\mu}.$$

$$\mathbb{E}[X^2] = \int x^2 f_X d\lambda = \int_0^{\infty} x^2 \mu e^{-\mu x} dx = \frac{2}{\mu^2}.$$

Example 2: Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Find $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.
Solution: Recall that the density of X is given by $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Thus, we have

$$\mathbb{E}[X] = \int x f_X d\lambda = \int_{-\infty}^{\infty} x \ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mu.$$

$$\mathbb{E}[X^2] = \int x^2 f_X d\lambda = \int_{-\infty}^{\infty} x^2 \ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mu^2 + \sigma^2.$$

Example 3: Let X be a one-sided Cauchy random variable i.e. $f_X(x) = \frac{2}{\pi}\frac{1}{1+x^2}$ for $x \geq 0$. Find $\mathbb{E}[X]$.
Solution: We have

$$\mathbb{E}[X] = \int x f_X d\lambda = \int_0^\infty x \; \frac{2}{\pi}\frac{1}{1+x^2} dx = \infty.$$

Example 4: Let X be a two-sided Cauchy random variable i.e., $f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ for $\forall x \in \mathbb{R}$. Find $\mathbb{E}[X]$.
Solution: In this case the random variable X takes both positive and negative values. Hence, we need to find $\mathbb{E}[X_+]$ and $\mathbb{E}[X_-]$ seperately and then evaluate $\mathbb{E}[X] = \mathbb{E}[X_+] - \mathbb{E}[X_-]$. Recall that $X_+ = \max(X, 0)$ and $X_- = -\min(X, 0)$. Thus,

$$X_+(\omega) = 0 \text{ for } \omega \in A = \{\omega \in \Omega | X(\omega) < 0\},$$
$$X_+(\omega) = X(\omega) \text{ for } \omega \in A^c.$$

Similarly,

$$X_-(\omega) = 0 \text{ for } \omega \in B = \{\omega \in \Omega | X(\omega) > 0\},$$
$$X_-(\omega) = -X(\omega) \text{ for } \omega \in B^c.$$

It is easy to see that $\mathbb{P}(A) = \mathbb{P}(B) = 0.5$. Next, we have

$$\mathbb{E}[X_+] = \int x d\mathbb{P}_{X_+}$$
$$\mathbb{E}[X_+] = 0 \times \mathbb{P}(A) + \int_0^\infty x \; \frac{1}{\pi}\frac{1}{1+x^2} dx = \infty.$$

Similarly,

$$\mathbb{E}[X_-] = \int x d\mathbb{P}_{X_-}$$
$$\mathbb{E}[X_-] = 0 \times \mathbb{P}(B) + \int_{-\infty}^0 -x \; \frac{1}{\pi}\frac{1}{1+x^2} dx = \infty.$$

Thus, we have a case of $\infty - \infty$ and $\mathbb{E}[X]$ is undefined.

Note that in Example 2 also, X takes both positive and negative values and we should find $\mathbb{E}[X_+]$ and $\mathbb{E}[X_-]$ seperately and evaluate $\mathbb{E}[X] = \mathbb{E}[X_+] - \mathbb{E}[X_-]$. But in that case both $\mathbb{E}[X_+]$ and $\mathbb{E}[X_-]$ are finite, allowing us to integrate with respect to the pdf $f_X(x)$ from $-\infty$ to ∞ directly.

Note: For the two sided Cauchy,

$$\int_{-\infty}^{\infty} \frac{1}{\pi}\frac{1}{1+x^2} dx \;\triangleq\; \lim_{\substack{M_1 \to -\infty \\ M_2 \to \infty}} \int_{M_1}^{M_2} \frac{1}{\pi}\frac{1}{1+x^2} dx.$$

The above limit does not exist and hence the integral is not defined.

21.2 Fatou's Lemma

Before we state Fatou's lemma, let us motivate it with an elementary result.

Lemma 21.2 *Let X and Y be random variables. Then,*

$$\mathbb{E}\left[\min(X,Y)\right] \leq \min\left(\mathbb{E}[X], \mathbb{E}[Y]\right).$$

$$\mathbb{E}\left[\max(X,Y)\right] \geq \max\left(\mathbb{E}[X], \mathbb{E}[Y]\right).$$

Proof: By definition, we have

$$\min(X,Y) \leq X.$$
$$\min(X,Y) \leq Y.$$

Taking expectations on both the sides,

$$\mathbb{E}\left[\min(X,Y)\right] \leq \mathbb{E}[X].$$
$$\mathbb{E}\left[\min(X,Y)\right] \leq \mathbb{E}[Y].$$

Combining the above two equations, we get

$$\mathbb{E}\left[\min(X,Y)\right] \leq \min\left(\mathbb{E}[X], \mathbb{E}[Y]\right).$$

The other statment of the lemma involving maximum of X and Y can be proved in a similar way and is left to the reader as an exercise.

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The above lemma can be generalized to any finite collection of random variables and a similar proof can be worked out. Fatou's Lemma generalizes this idea for a sequence of random variables.

Lemma 21.3 ***Fatou's Lemma:*** *Let Y be a random variable that satisfies $\mathbb{E}[|Y|] < \infty$. Then the following holds,*

- *If $Y \leq X_n$, for all n, then $\mathbb{E}\left[\liminf_{n\to\infty} X_n\right] \leq \liminf_{n\to\infty} \mathbb{E}[X_n]$.*
- *If $Y \geq X_n$, for all n, then $\mathbb{E}\left[\limsup_{n\to\infty} X_n\right] \geq \limsup_{n\to\infty} \mathbb{E}[X_n]$.*

Proof: Let us start by proving the first statement. For some n we have

$$\inf_{k\geq n} X_k - Y \leq X_m - Y, \qquad \forall\ m \geq n.$$

Taking expectations,

$$\mathbb{E}\left[\inf_{k\geq n} X_k - Y\right] \leq \mathbb{E}[X_m - Y], \qquad \forall\ m \geq n.$$

Taking infimum with respect to m on R.H.S, we obtain

$$\mathbb{E}\left[\inf_{k\geq n} X_k - Y\right] \leq \inf_{m\geq n} \mathbb{E}[X_m - Y], \qquad \forall\ m \geq n.$$

Let $Z_n = \inf_{k\geq n} X_k - Y$. Note that $Z_n \geq 0$ since $X_m \geq Y\ \forall\ m$ and Z_n is a non-decreasing sequence of random variables.

Also, $Z = \lim_{n\to\infty} Z_n = \liminf_{n\to\infty} X_n - Y$. By MCT, we have

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n - Y\right] \leq \liminf_{n\to\infty} \mathbb{E}[X_n - Y].$$

As $\mathbb{E}[|Y|] < \infty$, we can invoke linearity of expectation to get the first result of Fatou's lemma.

The second statement can be proved similarly and is left to the reader as an exercise.

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21.3 Dominated Convergence Theorem

The DCT is an important result which asserts a sufficient condition under which we can interchange a limit and integral.

Theorem 21.4 *Consider a sequence of random variables X_n that converges almost surely to X. Suppose there exists a random variable Y such that $|X_n| \leq Y$ almost surely for all n and $\mathbb{E}[Y] < \infty$. Then, we have*

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

Proof: We have $|X_n| \leq Y$ which implies $-Y \leq X_n \leq Y$. We can now apply Fatou's lemma to obtain

$$\mathbb{E}[X] = \mathbb{E}\left[\liminf_{n\to\infty} X_n\right] \leq \liminf_{n\to\infty} \mathbb{E}[X_n] \leq \limsup_{n\to\infty} \mathbb{E}[X_n] \leq \mathbb{E}\left[\limsup_{n\to\infty} X_n\right] = \mathbb{E}[X].$$

Thus, all the inequalities in the above equation must be met with equalities and we have

$$\mathbb{E}[X] = \liminf_{n\to\infty} \mathbb{E}[X_n] = \limsup_{n\to\infty} \mathbb{E}[X_n],$$

which proves that the limit, $\lim_{n\to\infty} \mathbb{E}[X_n]$ exists and is given by

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

Thus we see that Dominated Convergence theorem (DCT) is a direct consequence of Fatou's Lemma. The name "dominated" is intuitive because we need $|X_n|$ to be bounded by some random variable Y almost surely for every n. However, we do not require X_n's to be monotonically increasing as in the case of MCT.

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Corollary 21.5 *A special case of DCT is known as Bounded Convergence theorem (BCT). Here, the random variable Y is taken to be a constant random variable. BCT states that if there exists a constant $c \in \mathbb{R}$ such that $|X_n| \leq c$ almost surely for all n, then $\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X]$.*

21.4 Exercise

1. *[MIT OCW problem set]* A workstation consists of three machines, M_1, M_2 and M_3 , each of which will fail after an amount of time T_i which is an independent exponentially distributed random variable, with parameter 1. Assume that the times to failure of the different machines are independent. The workstation fails as soon as both of the following have happened:

   (a) Machine M_1 has failed.

   (b) Atleast one of the machines M_2 or M_3 has failed.

   Find the expected value of the time to failure of the workstation.

2. *[Assignment problem, University of Cambridge]* Let Z be an exponential random variable with parameter $\lambda = 1$ and $Z_{int} = \lfloor Z \rfloor$. Compute $\mathbb{E}[Z_{int}]$.

3. *[Prof. Pollak, Purdue University]* Suppose S_k and S_n are the prices of a financial instrument on days k and n, respectively. For $k < n$, the gross return $G_{k,n}$ between days k and n is defined as $G_{k,n} = \frac{S_n}{S_k}$ and is equal to the amount of money you would have on day n if you invested \$1 on day k. Let $G_{k,k+1}$ be lognormal random variable with parameters μ and σ^2, $\forall k \geq 1$, and the random variables $G_{j,j+1}$ and $G_{k,k+1}$ are independent and identically distributed $\forall k \neq j$. Find the expected total gross return from day 1 to day n.