EE 5110: Probability Foundations for Electrical Engineers | July-November 2015

Lecture 20: Expectation of Discrete RVs, Expectation over Different Spaces

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20.1 Expectations of Discrete RVs

A discrete random variable $X(\omega)$, (which only takes a countable set of values) can be represented as follows:

Definition 20.1 $X(\omega) = \sum_{i=1}^{\infty} a_i \mathbb{I}_{A_i}(\omega)$ *where* $X \geq 0$.

In the canonical representation, the a_i's are non-negative and distinct, and the A_i's are disjoint. It is easy to see that the A_i's partition the sample space. Let us now define a sequence of simple random variables, which approximate X from below.

Definition 20.2 *Define* $X_n(\omega) = \sum_{i=1}^{n} a_i \mathbb{I}_{A_i}(\omega)$.

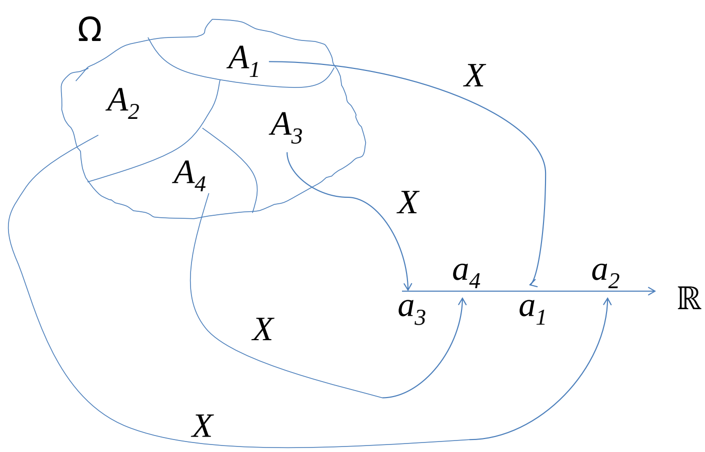

Figure 20.1: Simple random variable

Note that $\forall \omega$, $X_n(\omega) \leq X_{n+1}(\omega)$, where $n \geq 1$. Next, let us fix $\omega \in \Omega$. Since A_i's partition Ω, there exists $k \geq 1$ such that $\omega \in A_k$. Thus, $\forall n \geq k$, $X_n(\omega) = a_k$ and $\forall n < k$, $X_n(\omega) = 0$. Therefore,

$$\lim_{n \to \infty} X_n(\omega) = X(\omega) \qquad \forall \omega \in \Omega. \tag{20.1}$$

In other words, $X_n(\omega)$ is a sequence of simple functions converging monotonically to $X(\omega)$. Now applying the Monotone Convergence Theorem (MCT) to the sequence of random variables X_n,

$$\begin{aligned}
\mathbb{E}[X] &= \lim_{n\to\infty} \mathbb{E}[X_n], \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} a_i \mathbb{P}(A_i), \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} a_i \mathbb{P}(X = a_i), \\
\Rightarrow \mathbb{E}[X] &= \sum_{i=1}^{\infty} a_i \mathbb{P}(X = a_i). \qquad (20.2)
\end{aligned}$$

The limit of the sum is well-defined as X is a non-negative random variable and it either converges to some positive real number or goes to $+\infty$. If X is discrete but takes on both positive and negative values, we write $X = X_+ - X_-$, where $X_+ = \max(X, 0)$ and $X_- = -\min(X, 0)$. Then, we compute

$$\mathbb{E}[X] = \mathbb{E}[X_+] - \mathbb{E}[X_-]. \qquad (20.3)$$

The above is meaningful when *at least one* of the expectation on the right hand side is finite. We now give some examples.

1. $X \sim$ Geometric(p) - $\mathbb{E}[X] = \sum\limits_{i=1}^{\infty} i(1-p)^{i-1}p = \frac{1}{p}$.
   This tells us that, for a geometric random variable, the expected number of trials for the first success to occur scales as $\frac{1}{p}$.

2. $\mathbb{P}(X = k) = \frac{6}{\pi^2}\frac{1}{k^2}$ for $k \geq 1$ - For this probability distribution,the expectation is calculated as

$$\mathbb{E}[X] = \frac{6}{\pi^2} \sum_{i=1}^{\infty} i \left(\frac{1}{i^2}\right) = +\infty. \qquad (20.4)$$

   In this example, we see that a random variable can have infinite expectation.

3. $\mathbb{P}(X = k) = \frac{3}{\pi^2}\frac{1}{k^2}$ for $k \in \mathbb{Z}/\{0\}$ - For this probability distribution, the expectation is calculated as $\mathbb{E}[X] = \mathbb{E}[X_+] - \mathbb{E}[X_-]$. However, both the expectations $\mathbb{E}[X_+]$ and $\mathbb{E}[X_-]$ are infinite! Therefore, $\mathbb{E}[X]$ is not defined! This is an example of a discrete random variable with undefined expectation.

20.2 Connection between Riemann and Lebesgue integrals

The connection is given by the following theorem which we state without proof.

Theorem 20.3 *Let f be measurable and Riemann integrable over an interval $[a, b]$. Then,*

$$\int_{[a,b]} f \, d\lambda \text{ exists, and } \int_{[a,b]} f \, d\lambda = \int_a^b f(x) \, dx. \qquad (20.5)$$

Here, λ is the Lebesgue measure on $\mathbb{R}$. The integral on the left is a Lebesgue integral while the one on the right is the standard Riemann integral.

20.3 Expectations on different spaces

We often want to compute the expectation of a function of a random variable, say $Y = f(X)$, where both X and Y are random variables and $f(\cdot)$ is a measurable function on $\mathbb{R}$. The following theorem asserts that the expectation can be computed over different spaces, to obtain the 'same answer.' For example, we can compute the expectation of Y by either working in the X-space or the Y-space to write (for discrete random variables)

$$\sum_i y_i \mathbb{P}(Y = y_i) = \sum_i f(a_i) \mathbb{P}(X = a_i), \tag{20.6}$$

where $y_i = f(a_i)$. This is just a special case of the following theorem

Theorem 20.4 *Denote the probability measure on the sample space by $\mathbb{P}$, on the range space of X as $\mathbb{P}_X$ and on range space of Y as $\mathbb{P}_Y$. Then, $\int Y \, d\mathbb{P} = \int f \, d\mathbb{P}_X = \int y \, d\mathbb{P}_Y$ where $Y = f(X)$ and the integrals are over the respective spaces.*

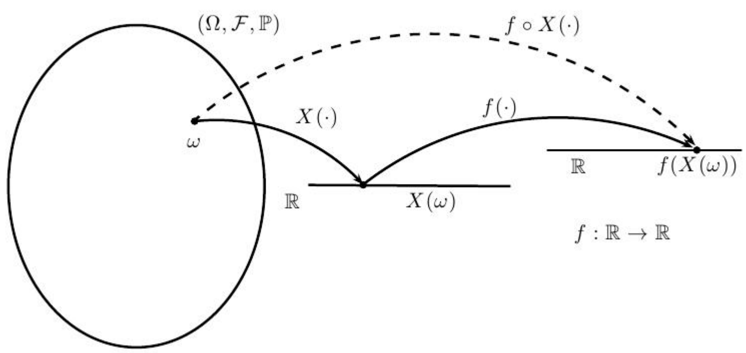

Figure 20.2: Different spaces considered

Proof: Let f be a simple function which takes values $y_1, y_2 \cdots y_n$. Then,

$$\begin{aligned} \int Y \, d\mathbb{P} &= \sum_{i=1}^{n} y_i \mathbb{P}(\omega | Y(\omega) = y_i), \\ &= \sum_{i=1}^{n} y_i \mathbb{P}(\omega | f(X(\omega)) = y_i). \end{aligned}$$

Now, looking at the second integral, we have

$$\begin{aligned}
\int f\, d\mathbb{P}_X &= \sum_{i=1}^{n} y_i \mathbb{P}_X(x \in \mathbb{R} | f(x) = y_i), \\
&= \sum_{i=1}^{n} y_i \mathbb{P}_X(f^{-1}(y_i)), \\
&= \sum_{i=1}^{n} y_i \mathbb{P}(\omega | \omega : X(\omega) \in f^{-1}(y_i)), \\
&= \sum_{i=1}^{n} y_i \mathbb{P}(\omega | f(X(\omega)) = y_i).
\end{aligned}$$

Now, we extend the above to the case when f is a non-negative measurable function. Let $\{f_n\}$ be a sequence of simple functions such that $f_n \uparrow f$ according to the construction given in the previous lecture. Thus, $f_n(X) \uparrow f(X)$ and,

$$\begin{aligned}
\int Y\, d\mathbb{P} &= \int (f \cdot X)\, d\mathbb{P}, \\
&= \int f(X)\, d\mathbb{P}, \\
&= \lim_{n\to\infty} \int f_n(X)\, d\mathbb{P} \qquad \text{(by MCT)}, \\
&= \lim_{n\to\infty} \int f_n\, d\mathbb{P}_X \qquad (\because \text{simple function}), \\
&= \int f\, d\mathbb{P}_X \qquad \text{(by MCT)}.
\end{aligned}$$

This can now be simply extended to the case where g takes both negative and positive values. ■

A simple corollary of this theorem is that $\int X\, d\mathbb{P} = \int x\, d\mathbb{P}_X$.

20.4 Exercise

1. *[Dimitri P.Bertsekas]* Let X be a random variable with PMF $p_X(x) = \frac{x^2}{a}$, if $x = -3, -2, -1, 0, 1, 2, 3$ and zero otherwise. Compute a and $\mathbb{E}[X]$.
2. *[Dimitri P.Bertsekas]* As an advertising campaign, a chocolate factory places golden tickets in some of its candy bars, with the promise that a golden ticket is worth a trip through the chocolate factory, and all the chocolate you can eat for life. If the probability of finding a golden ticket is p, find the expected number of bars you need to eat to find a ticket.
3. *[Dimitri P.Bertsekas]* On a given day, your golf score takes values from the range 101 to 110, with probability 0.1, independent of other days. Determined to improve your score, you decide to play on three different days and declare as your score the minimum X of the scores X_1, X_2 and X_3 on the different days. By how much has your expected score improved as a result of playing on three days?
4. *[Papoulis]* A biased coin is tossed and the first outcome is noted. Let the probability of head occuring be p and that of a tail be $q = 1 - p$. The tossing is continued until the outcome is the complement of the first outcome, thus completing the first run. Let X denote the length of the first run. Find the PMF of X and show that $\mathbb{E}[X] = \frac{p}{q} + \frac{q}{p}$.