Probability Foundations for Electrical Engineers **July-November 2015**

Lecture 1: Basic Set Theory

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We will begin with an informal and intuitive approach to set theory known as "Naive Set Theory".

1.1 What is a set?

A set can be thought of as a collection of *well-defined* objects. By well-defined, we mean that an object either belongs to a set or it does not. Objects belonging to a set are known as *elements* of the set. Sets can be specified in 2 ways:

1. *Extensional definition*-All the elements of the set are listed out explicitly and enclosed within curly brackets. E.g., the set of all natural numbers from 1 to 5 may be specified as $A = \{1, 2, 3, 4, 5\}$.
2. *Intensional definition*-Here, a set is defined in terms of the property which is satisfied by all its members. This is also known as the *set builder notation.* E.g., the set A above may also be defined as $A = \{x|x : x \leq 5, x \in \mathbb{N}\}$. In general, some set C may defined as $C = \{x|P(x)\}$, where $P(x)$ is some property.

We now define the notion of a subset and use the idea of subsets to define when two sets are identical.

Definition 1.1 *(i) A set A is said to be a subset of (or contained in) another set B if every element of A is also an element of B. This is denoted as $A \subseteq B$. Here, B is said to be a superset of A.*

(ii) A is a proper *subset of B (denoted $A \subset B$) if A is a subset of B and there is at least one element in B which does not belong to A.*

(iii) Two sets A and B are said to be identical (or equal) if $A \subseteq B$ and $B \subseteq A$. In other words, every element of A is an element of B, and vice versa.

Two special sets of interest are:

1. The *universal set* U, a set which contains all elements[1]
2. The *empty set* $\emptyset$, which has, as its name indicates, no elements. It is a subset of every set including itself and a proper subset of every set excluding itself.

1.2 Operations on sets

1.2.1 Complement

Taking the complement of a set is a unary operation (i.e., only one set is operated upon) defined as

[1]However, in the usual formulations of set theory, the concept of a universal set leads to a paradox known as Russell's paradox. The interested student may look up this famous paradox,'$en.wikipedia.org/wiki/Russell's\_paradox$'.

Definition 1.2 *For a set A, its complement is defined as* $A^c \triangleq \{x | x \notin A, x \in U\}$.

The context for the complement of a set is provided by the universal set U. The Venn diagram representation of a set's complement is

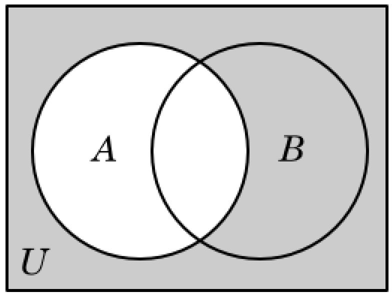

Figure 1.1: Complement (gray area) of a set A

1.2.2 Union and Intersection

Let $\mathcal{I}$ be an abstract index set. Consider a family of sets $\{A_i,\ i \in \mathcal{I}\}$ indexed by $\mathcal{I}$.

Definition 1.3 Union*: The union of* $\{A_i,\ i \in \mathcal{I}\}$ *is defined as*

$$\bigcup_{i \in \mathcal{I}} A_i = \{x | x \in A_j \text{ for some } j \in \mathcal{I}\}.$$

In words, the union $\bigcup_{i \in \mathcal{I}} A_i$ is a set consisting of those elements which are elements of *at least one* of the A_i's.

Definition 1.4 Intersection*: The intersection of* $\{A_i,\ i \in \mathcal{I}\}$ *is defined as*

$$\bigcap_{i \in \mathcal{I}} A_i = \{x | x \in A_j \text{ for every } j \in \mathcal{I}\}.$$

In words, the intersection $\bigcap_{i \in \mathcal{I}} A_i$ is a set consisting of those elements which are elements of *all* the A_i's.

Remark: 1.5 *When the index set $\mathcal{I}$ is a finite set, say $\mathcal{I} = \{1, 2, 3\}$ the definition of union given above coincides with the "middle-school" understanding of unions, i.e., taking the union of sets one-at-a-time. For example, $\bigcup_{i=1}^{3} A_i = A_1 \bigcup A_2 \bigcup A_3$. However, this "one-by-one" interpretation completely breaks down when the index set $\mathcal{I}$ is infinite. For example when $\mathcal{I} = \mathbb{N}$, the union $\bigcup_{i=1}^{\infty} A_i$ does not have any interpretation in terms of taking unions one by one, till infinity. After all, there is no A_∞ in the family $\{A_i, i \in \mathbb{N}\}$, and there is no notion of "limiting unions". Thus, $\bigcup_{i=1}^{\infty} A_i$ should be interpreted just as Definition 1.3 says: it is the set of all elements contained in at least one of the $A_i,\ i \in \mathbb{N}$.*

In order to avoid the (dangerous) temptation to interpret $\bigcup_{i=1}^{\infty} A_i$ as some sort of a limit of finite, "one-by-one" unions, a better notation would be to use $\bigcup_{i \in \mathbb{N}} A_i$, instead of the potentially misleading but more commonly used notation $\bigcup_{i=1}^{\infty} A_i$.

The following useful identities related to unions and intersections can be proven easily (*do it!*) from the definitions.

$$\left(\bigcap_{i\in\mathcal{I}} A_i\right)\bigcup B = \bigcap_{i\in\mathcal{I}}(A_i \bigcup B), \tag{1.1}$$

and

$$\left(\bigcup_{i\in\mathcal{I}} A_i\right)\bigcap B = \bigcup_{i\in\mathcal{I}}(A_i \bigcap B), \tag{1.2}$$

An especially important set of laws regarding the interchangeability of unions and intersections under the complement operation are *De Morgan's laws.* The two laws are (prove them!):

1. $(\bigcap_{i\in\mathcal{I}} A_i)^c = \bigcup_{i\in\mathcal{I}} A_i^c$, that is, the complement of the intersection is the union of the complements.
2. $(\bigcup_{i\in\mathcal{I}} A_i)^c = \bigcap_{i\in\mathcal{I}} A_i^c$, that is, the complement of the union is the intersection of the complements.

Finally, the relative complement operation on two sets allows us to "subtract" one from the other.

Definition 1.6 Relative complement*: The relative complement of B in A is defined as $A\backslash B \triangleq \{x|x \in A, x \notin B\} = A \cap B^c$. Similarly, the relative complement of A in B is defined as $B\backslash A \triangleq \{x|x \in B, x \notin A\} = B \cap A^c$.*

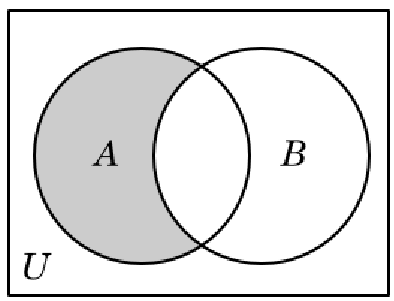

Figure 1.2: Relative complement of B in A

The unary complement operation for a set A can also be understood as the relative complement of A in U, the universal set.

1.2.3 Cartesian products

A Cartesian product is an operation on sets which returns a product set from multiple sets.

Definition 1.7 Cartesian product*: The Cartesian product of 2 sets A and B is defined as $A \times B \triangleq \{(x, y) : x \in A, y \in B\}$, that is, it is the set of all ordered pairs of elements from the two sets, such that the first component belongs to A and the second to B.*

For example, if $A = \{1, 2\}$ and $B = \{a\}$, then $A \times B = \{(1, a), (2, a)\}$ and $B \times A = \{(a, 1), (a, 2)\}$. Clearly, this operation is not commutative. The Cartesian product of n sets $A_1, A_2 \cdots A_n$ is

$$A_1 \times A_2 \cdots \times A_n = \{(a_1, a_2 \cdots, a_n) : a_i \in A_i\}$$

If all the n sets are identical, then we get

$$A^n = \{(a_1, a_2 \cdots, a_n) : a_i \in A\}$$

1.3 Power sets

Definition 1.8 Power set*: The power set of a set A, denoted as $\mathcal{P}(A)$ or 2^A, is the set of all subsets of A including the null set $\emptyset$ and A itself.*

For example, the power set of $A = \{1, 2\}$ is

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

A power set is an example of a class, which is a collection of sets and is usually denoted by a script letter, like so: $\mathcal{A}$. The union and intersection operations extend to classes, as does the idea of subsets, in a suitably modified form.

1.4 Functions

Definition 1.9 *A function f from a set A to another set B is a subset of the Cartesian product $(A \times B)$ of the sets such that every element of A is the first component of one and only one ordered pair in the subset. In simple terms, it is a rule that maps every element from set A to a unique element in set B. It is commonly denoted as $f : A \to B$ and A is known as the* domain *while B is known as the* codomain.

The element in the codomain (say, b) which is associated with an element in the domain (say, a) is known as the *image* of the element 'a', and 'a' by itself is called the *argument* of the function 'f' and is also termed as *pre-image* of the element 'b'. Then, we say f maps a to b and is represented as $b = f(a)$. The *range* of a function is the set of all elements in the co-domain which are images for elements in the domain, hence, it is the subset (not necessarily proper subset) of the codomain . Functions can be classified as follows:

1. *Injective*: An injective or one-to-one function is one where $a \neq b \Rightarrow f(a) \neq f(b), \forall a, b \in \text{domain}(f)$. E.g., function $f : \mathbb{N} \to \mathbb{R}$ defined as $f(x) = x$, $\forall x \in \mathbb{N}$, is an injective function.

2. *Surjective*: A surjective or onto function is one where $\forall b \in \text{codomain}(f), \exists a \in \text{domain}(f)$ *such that* $f(a) = b$. For example, the following are surjective functions:
   (i) Let $A = \{1, 2, 3\}$ and $B = \{0, 1\}$. The function $g : A \to B$ defined as $g(1) = 0, g(2) = 0$ and $g(3) = 1$ is a surjective function.
   (ii) The function $h : \mathbb{R} \to \mathbb{R}$ defined as $h(x) = x$, $\forall x \in \mathbb{R}$ is also surjective.

A function which is both injective and surjective is known as a bijective function (or a *bijection*). The example of function 'h' stated above is also a bijective function. An inverse function can be defined for a bijection since the mapping is unique and the entire codomain is covered. The notion of a bijection can be used to understand the equicardinality of infinite sets, i.e., when can we say that the "size" of two infinte sets is equal? This question will be answered in the subsequent lectures.