EE5110 : Probability Foundations for Electrical Engineers **July-November 2015**

Lecture 19: Monotone Convergence Theorem

Lecturer: Dr. Krishna Jagannathan *Scribes: Vishakh Hegde*

In this lecture, we present the Monotone Convergence Theorem (henceforth called MCT), which is considered one of the cornerstones of integration theory. The MCT gives us a sufficient condition for interchanging limit and integral. We also prove the linearity property of integrals using the MCT. Recall the $g_n \longrightarrow g$ μ.a.e. if $g_n(\omega) \longrightarrow g(\omega)$ $\forall \omega \in \Omega$ except possibly on a set of $\mu-$measure zero.

19.1 Monotone Convergence Theorem

Theorem 19.1 *Let $g_n \geq 0$ be a sequence of measurable functions such that $g_n \uparrow g$ μ.a.e. (That is, except perhaps on a set of μ-measure zero, we have $g_n(\omega) \rightarrow g(\omega)$, and $g_n(\omega) \leq g_{n+1}(\omega)$, $n \geq 1$). We then have $\int g_n d\mu \uparrow \int g d\mu$. In other words,*

$$\lim_{n\to\infty} \int g_n \, d\mu = \int g \, d\mu.$$

See Section 5.2 in Lecture 11 of [1] for the proof.

Example 19.2 *Consider $([0,1], \mathcal{B}, \lambda)$ and consider the sequence of functions given by,*

$$f_n(\omega) = \begin{cases} n, & \text{if } 0 < \omega \leq 1/n, \\ 0, & \text{otherwise.} \end{cases}$$

$$\int f_n d\lambda = 1, \forall n \Rightarrow \lim_{n\to\infty} \int f_n d\lambda = 1.$$

For $\omega > 0$, we have,

$$\lim_{n\to\infty} f_n(\omega) = 0.$$

For $\omega = 0$, we have,

$$\lim_{n\to\infty} f_n(\omega) = \infty.$$

Therefore we have,

$$\int f d\lambda = 0.$$

Hence we see that,

$$\int f d\lambda \neq \lim_{n\to\infty} \int f_n d\lambda.$$

Note that monotonicity does not hold in this example.

19.2 Linearity of Integrals

In this section, we will prove the linearity property of integrals, using the MCT. Recall that we stated the linearity property in the previous lecture as **PAI 4** but proved it only for simple functions. Here we prove it in full generality.

Let f and g be simple functions. Therefore we can express them as,

$$f = \sum_{i=1}^{n} a_i \mathbb{I}_{A_i},$$

$$g = \sum_{j=1}^{m} b_j \mathbb{I}_{B_j}.$$

Here A_i and B_i are $\mathcal{F}$ measurable sets and I_{A_i} and I_{B_j} are indicator variables. Summing f and g, we obtain,

$$f + g = \sum_{i=1}^{n}\sum_{j=1}^{m}(a_i + b_j)\mathbb{I}_{A_i \cap B_j}. \tag{19.1}$$

Note that f and g are canonical representations. This implies that A_i's are disjoint sets, and so are B_j's. Therefore $A_i \cap B_j$ are disjoint sets. Hence we have,

$$\begin{aligned}\int f + g \; d\mu &= \sum_{i=1}^{n}\sum_{j=1}^{m}(a_i + b_j)\mu(A_i \cap B_j),\\ &= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \mu(A_i \cap B_j) + \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \mu(A_i \cap B_j).\end{aligned}$$

By finite additivity property, we have,

$$\begin{aligned}\int f + g \; d\mu &= \sum_{i=1}^{n} a_i \mu(A_i) + \sum_{j=1}^{m} b_j \mu(B_j),\\ &= \int f \; d\mu + \int g \; d\mu.\end{aligned}$$

Next, we need to prove linearity for non-negative measurable functions. Let f_n and g_n (with $n \geq 1$) be sequences of simple functions where, $f_n \uparrow f$ and $g_n \uparrow g$. Such a simple sequence always exist for every non-negative measurable function, as we will show in the next section. Now, since f_n and g_n are monotonic, $f_n + g_n$ is monotonic. Then we can show that $(f_n + g_n) \uparrow (f + g)$. Using MCT, we have,

$$\int (f + g) d\mu = \lim_{n\to\infty} \int (f_n + g_n) d\mu. \tag{19.2}$$

But f_n and g_n are simple functions. We know that, for simple functions,

$$\int (f_n + g_n) d\mu = \int f_n d\mu + \int g_n d\mu.$$

Thus,

$$\begin{aligned}\lim_{n\to\infty} \int (f_n + g_n) d\mu &= \lim_{n\to\infty} \int f_n d\mu + \lim_{n\to\infty} \int g_n d\mu,\\ &\overset{MCT}{=} \int f d\mu + \int g d\mu.\end{aligned}$$

This implies that,

$$\int (f+g)d\mu = \int f d\mu + \int g d\mu. \tag{19.3}$$

This proves linearity for non-negative functions.

For arbitrary measurable functions f and g, we can write them as $f = f_+ - f_-$ and $g = g_+ - g_-$ where f_+, f_-, g_+ and g_- are non-negative measurable functions. A similar proof can then be worked out which completes the proof of linearity.

19.3 Integration using simple functions

Our earlier definition $\int g d\mu = \sup_{q \in S(g)} \int q d\mu$ helped us to prove some properties of abstract integrals quite easily. However, it does not give us a practical way of performing the integration. In this section, we present a method to explicitly compute the integral, using the MCT. First, we approximate the function to be integrated using simple functions from below. Specifically, define

$$g_n(\omega) = \begin{cases} n, & \text{if } g(\omega) \geq n, \\ \frac{i}{2^n}, & \text{if } \frac{i}{2^n} \leq g(\omega) < \frac{i+1}{2^n}; i \in \{0, 1, \dots, n2^n - 1\}. \end{cases} \tag{19.4}$$

Thus, the function to be integrated in quantized to $n2^n$ levels. Next, we note here that $g_n(\omega)$ is a simple function since it can be written as

$$g_n(\omega) = \sum_{i=0}^{n2^n-1} \frac{i}{2^n} \mathbb{I}_{\{\omega : \frac{i}{2^n} \leq g(\omega) < \frac{i+1}{2^n}\}} + n\mathbb{I}_{\{g_n(\omega) \geq n\}}. \tag{19.5}$$

Claim 1: We can easily show that:

- $g_n(\omega) \to g(\omega)\ \forall \omega \in \Omega$.
- $g_n(\omega) \leq g_{n+1}(\omega)\ \forall \omega \in \Omega$ and $\forall n \in \mathbb{N}$.

Therefore, using MCT, we have,

$$\begin{aligned} \int g d\mu &= \lim_{n\to\infty} \int g_n d\mu, \\ &= \lim_{n\to\infty} \sum_{i=0}^{n2^n-1} \frac{i}{2^n} \mu\left(\omega : \frac{i}{2^n} \leq g(\omega) \leq \frac{i+1}{2^n}\right) + n\mu\left(g_n(\omega) \geq n\right). \end{aligned}$$

Now, if g is bounded the second term $\mu(g_n(\omega) \geq n)$ will be 0 and if g is unbounded, it may or may not be finite.

This gives us an explicit way to compute the abstract integral.

19.4 Exercise:

1. Prove Claim 1.

2. Let X be a *non-negative* random variable (not necessarily discrete or continuous) with $\mathbb{E}[X] < \infty$.

   (a) Prove that $\lim_{n\to\infty} n\mathbb{P}(X > n) = 0$. [Hint: Write $\mathbb{E}[X] = \mathbb{E}[X\mathbb{I}_{\{X\leq n\}}] + \mathbb{E}[X\mathbb{I}_{\{X>n\}}]$.]

   (b) Prove that $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)\, \mathrm{d}x$. Yes, the integral on the right *is* just a plain old Riemann integral! [Hint: Write out $\mathbb{E}[X] = \int x \, \mathrm{d}\mathbb{P}_X$ as the limit of a sum, and use part (a) for the last term.]

   We say a random variable X is stochastically larger than a random variable Y, and denote by $X \geq_{st} Y$, if $\mathbb{P}(X > a) \geq \mathbb{P}(Y > a)\ \forall a \in \mathbb{R}$.

   (c) For non-negative random variables X and Y, show that if $X \geq_{st} Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$.

3. Show that $f(x) = x^{-\alpha}$ is integrable on $[0, \infty)$ for $\alpha > 1$.

19.5 References:

[1] David Gamarnick and John Tsitsiklis, “Introduction to Probability”, *MIT OCW*, 2008.