EE5110: Probability Foundations for Electrical Engineers **July-November 2015**

Lecture 16: General Transformations of Random Variables

Lecturer: Dr. Krishna Jagannathan *Scribe: Ajay and Jainam*

In the previous lectures, we have seen few elementary transformations such as sums of random variables as well as maximum and minimum of random variables. Now we will look at general transformations of random variables. The motivation behind transformation of a random variable is illustrated by the following example. Consider a situation where the velocity of a particle is distributed according to a random variable V. Based on a particular realisation of the velocity, there will be a corresponding value of kinetic energy E and we are interested in the distribution of kinetic energy. Clearly, this is a scenario where we are asking for the distribution of a new random variable, which depends on the original random variable through a transformation. Such situations occur often in practical applications.

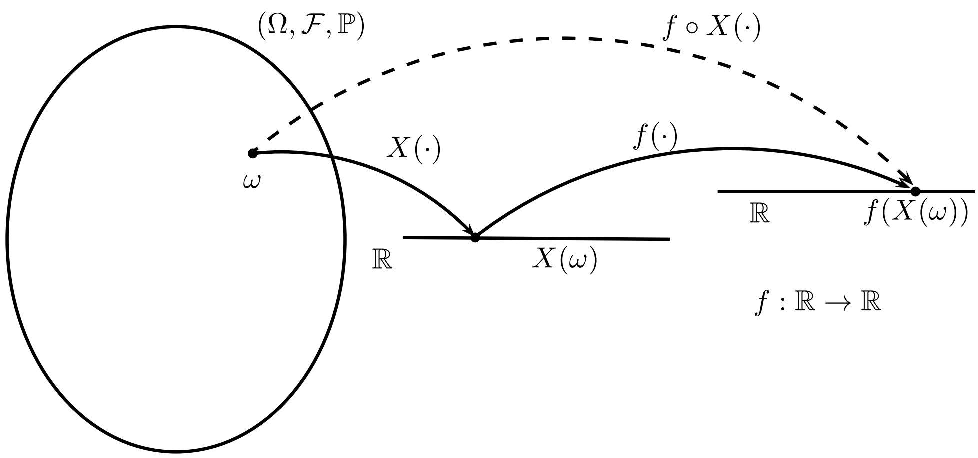

Figure 16.1: Transformation of random variable

16.1 Transformations of a Single Random Variable

Consider a random variable $X : \Omega \to \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Then $Y = g(X)$ is also a random variable and we wish to find the distribution of Y. Specifically, we are interested in finding the CDF $F_Y(y)$ given the CDF $F_X(x)$.

$$F_Y(y) = \mathbb{P}(g(X) \le y) = \mathbb{P}(\{\omega | g(X(\omega)) \le y\}).$$

Let B_y be the set of all x such that $g(x) \le y$. Then $F_Y(y) = \mathbb{P}_X(B_y)$.

We now illustrate this with the help of an example.

Example 1: *Let X be a Gaussian random variable of mean 0 and variance 1* i. e. $X \sim \mathcal{N}(0,1)$. *Find the distribution of $Y = X^2$.*

Solution:

$$F_Y(y) \; = \; \mathbb{P}(X^2 \leq y) \; = \; \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = \int\limits_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}),$$

where Φ *is the CDF of* $\mathcal{N}(0,1)$.
Now,

$$f_Y(y) = \frac{dF_Y(y)}{dy}.$$

From above,

$$F_Y(y) = \int\limits_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt = 2 \times \int\limits_{0}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt.$$

Changing variables $t^2 = u$*, we get*

$$F_y(y) = 2 \times \int\limits_{0}^{y} \frac{1}{2\sqrt{2\pi u}} e^{\frac{-u}{2}} du.$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{\frac{-y}{2}}, \quad \textit{for} \quad y > 0.$$

Note:

1. The random variable Y can take only non-negative values as it is square of a real valued random variable.
2. The distribution of square of the Gaussian random variable, $f_Y(y)$, is also known as Chi-squared distribution.

Thus, we see that given the distribution of a random variable X, the distribution of any function of X can be obtained by first principles. We now come up with a direct formula to find the distribution of a function of the random variable in the cases where the function is differentiable and monotonic.

Let X have a density $f_X(x)$ and g be a monotonically increasing function and let $Y = g(X)$. We then have

$$F_Y(y) = \mathbb{P}(Y \leq y) \; = \; \mathbb{P}(g(X) \leq y) \; = \int\limits_{-\infty}^{g^{-1}(y)} f_X(x) dx.$$

Note that as g is a monotonically increasing function, $g(x) \leq y \implies x \leq g^{-1}(y)$.

Let $x = g^{-1}(t)$, so $g'(x)dx = dt$.

$$F_Y(y) = \int\limits_{-\infty}^{y} f_X(g^{-1}(t)) \frac{dt}{g'(g^{-1}(t))}.$$

Differentiating, we get

$$f_Y(y) = f_X(g^{-1}(y))\frac{1}{g'(g^{-1}(y))}.$$

The second term on the right hand side of the above equation is referred to as the *Jacobian* of the transformation $g(\cdot)$.

It can be shown easily that a similar argument holds for a monotonically decreasing function g as well and we obtain

$$f_Y(y) = f_X(g^{-1}(y))\frac{-1}{g'(g^{-1}(y))}.$$

Hence, the general formula for distribution of monotonic functions of random variables is as under

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{\mid g'(g^{-1}(y)) \mid}. \tag{16.1}$$

Example 2: . *Let $X \sim \mathcal{N}(0,1)$. Find the distribution of $Y = e^X$.*

Solution: *Note that the function $g(x) = e^x$ is a differentiable, monotonically increasing function.*

As $g(x) = e^x$, we have $g^{-1}(y) = \ln(y)$ and $g'(g^{-1}(y)) = y$. Here we see that the Jacobian will be positive for all vaues of y and hence $\mid g'(g^{-1}(y)) \mid = g'(g^{-1}(y)) = y$.

Finally we have

$$f_Y(y) = \frac{f_X(\ln(y))}{y} = \frac{1}{y\sqrt{2\pi}}e^{\frac{-(\ln(y))^2}{2}} \quad \text{for} \quad y > 0.$$

This is the log-normal pdf.

Example 3: *Let $U \sim unif[0,1]$ i.e. U is a uniform random variable in the interval [0,1]. Find the distribution $Y = -\ln(U)$.*

Solution: *Note that $g(u) = -\ln(u)$ is a differentiable, monotonically decreasing function.*

As $g(u) = -\ln(u)$, we have $g^{-1}(y) = e^{-y}$ and $g'(g^{-1}(y)) = \frac{-1}{e^{-y}}$. Here we see that the Jacobian will be negative for all values of y and hence $\mid g'(g^{-1}(y)) \mid = -g'(g^{-1}(y)) = \frac{1}{e^{-y}}$.

Hence we have

$$f_Y(y) = \frac{f_U(e^{-y})}{\frac{1}{e^{-y}}} = \frac{1}{\frac{1}{e^{-y}}} = e^{-y} \text{ for } y \geq 0.$$

Note that Y is an exponential random variable with mean 1.

If X is a continous random variable with CDF $F_X(\cdot)$, then it can be shown that the random variable $Y = F(X)$ is uniformly distributed over $[0,1]$ (*see Exercise 2(a)*). It can be seen from this result that any continuous random variable Y can be generated from a uniform random variable $X \sim unif[0,1]$ by the transformation $Z = F_Y^{-1}(X)$ where $F_Y(\cdot)$ is the CDF of the random variable Y.

16.2 Transformation of Multiple Random Variables

Equation 16.1 can be extended to transformations of multiple random variables. Consider an n-tuple random variable $(X_1, X_2, \ldots, X_n)$ whose joint density is given by $f_{(X_1,X_2,\ldots,X_n)}(x_1, x_2, \ldots, x_n)$ and the

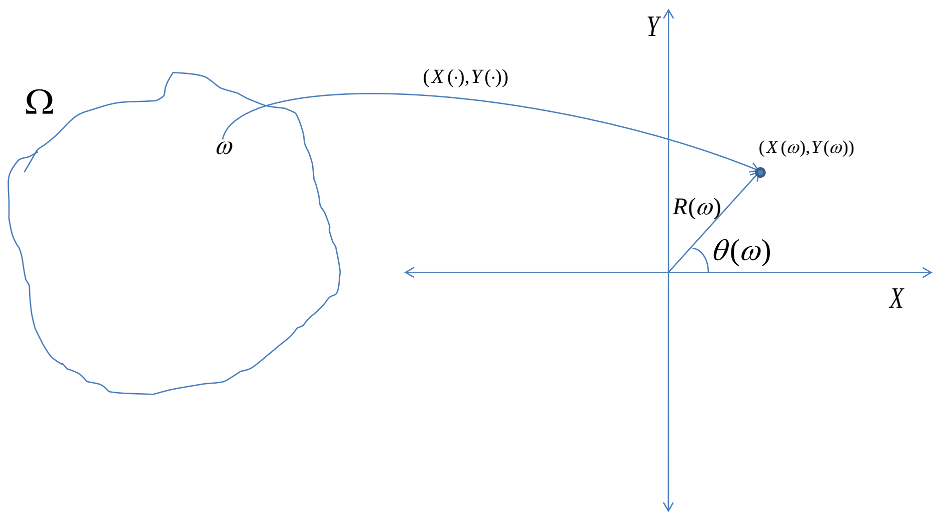

Figure 16.2: Mapping of a realization $(X(\omega), Y(\omega))$ to the polar co-ordinates $(R(\omega), \Theta(\omega))$

corresponding transformations are given by $Y_1 = g_1(X_1, X_2, \ldots, X_n), Y_2 = g_2(X_1, X_2, \ldots, X_n), \ldots, Y_n = g_n(X_1, X_2, \ldots, X_n)$. Succinctly, we denote this as a vector transformation $Y = g(X)$, where $g: \mathbb{R}^n \to \mathbb{R}^n$. We assume that the transformation g is invertible, and continuously differentiable. Under this assumption, the joint density of $f_{Y_1,Y_2,\ldots,Y_n}(Y_1, Y_2, \ldots, Y_n)$ is given by (see Section 2.2 in Lecture 10 of [1])

$$f_{Y_1,Y_2,\ldots,Y_n}(y_1, y_2, \ldots, y_n) = f_{(X_1,X_2,\ldots,X_n)}(g_1^{-1}(y))|J(y)|, \tag{16.2}$$

where $|J(y)|$ is the Jacobian matrix, given by

$$J(y) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} & . & . & . & \frac{\partial x_n}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} & . & . & . & \frac{\partial x_n}{\partial y_2} \\ . & . & & & & . \\ . & . & & \ddots & & . \\ . & . & & & & . \\ \frac{\partial x_1}{\partial y_n} & \frac{\partial x_2}{\partial y_n} & . & . & . & \frac{\partial x_n}{\partial y_n} \end{vmatrix}.$$

We now illustrate this with the help of an example.

Example 4: *Let the Euclidean co-ordinates of a particle be drawn from identically distributed independent Gaussian random variables of mean* 0 *and varinace* 1 *i.e.,* $X, Y \sim \mathcal{N}(0, 1)$. *Find the distribution of the particle's polar co-ordinates,* R *and* Θ.

Solution: *The corresponding transformations are given by* $X = R\cos\Theta$ *and* $Y = R\sin\Theta$.

Let us first evaluate the Jacobian. We have $x = r\cos\theta$ *and* $y = r\sin\theta$*. So we have* $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$ *and* $\frac{\partial y}{\partial \theta} = r\cos\theta$.

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r.$$

Next, we have $X, Y \sim \mathcal{N}(0,1)$ *and they are independent so*

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} \quad x, y \in \mathbb{R}.$$

From (16.2)*, we have*

$$f_{R,\Theta}(r,\theta) = \frac{1}{2\pi} e^{-\frac{(r\cos\theta)^2 + (r\sin\theta)^2}{2}} \times r \text{ where } r \geq 0 \text{ and } \theta \in [0, 2\pi].$$

The marginal densities of R *and* Θ *can be obtained from the joint distribution as given below.*

$$f_R(r) = \int_0^{2\pi} f_{R,\Theta}(r,\theta) d\theta = re^{-r^2/2}, \text{ for } r \geq 0.$$

$$f_\Theta(\theta) = \int_0^{\infty} f_{R,\Theta}(r,\theta) dr = \frac{1}{2\pi}, \text{ for } \theta \in [0, 2\pi].$$

The distribution $f_R(r)$ *is called the Rayleigh distribution, which is encountered quite often in Wireless Communications to model the gain of a fading channel. Note that the random variables* R *and* Θ *are independent since the joint distribution factorizes into the product of the marginals i.e.*

$$f_{R,\Theta}(r,\theta) = f_R(r) \times f_\Theta(\theta).$$

We now illustrate how transformations of random variables help us to generate random variables with different distributions given that we can generate only uniform random variables. Specifically, consider the case where all we can generate is a uniform random variable between 0 and 1 *i. e.* $unif[0,1]$ and we wish to generate random variables having Rayleigh, exponential and Guassian distribution.

Generate U_1 and U_2 as i.i.d. $unif[0,1]$. Next, let $\Theta = 2\pi U_1$ and $Z = -\frac{\ln(U_2)}{2}$. It can be verified that $\Theta \sim Unif[0, 2\pi]$ and $Z \sim Exp(0.5)$.

Thereafter, let $R = \sqrt{Z}$. It can be shown that R is a Rayleigh distributed random variable (*see Exercise 1*).

Lastly, let $X = R\cos\Theta$ and $Y = R\sin\Theta$. It is easy to see from Example 3 that X and Y will be i.i.d. $\mathcal{N}(0,1)$.

16.3 Exercises

1. Let $X \sim exp(0.5)$. Prove that $Y = \sqrt{X}$ is a Rayleigh distributed random variable.
2. (a) Let X be a random variable with a continuous distribution F.

(i) Show that the Random Variable $Y = F(X)$ is uniformly distributed over $[0, 1]$. [Hint: Although F is the distribution of X, regard it simply as a function satisfying certain properties required to make it a CDF !]

(ii) Now, given that $Y = y$, a random variable Z is distributed as Geometric with parameter y. Find the unconditional PMF of Z. Also, given $Z = z$ for some $z \geq 1, z \in \mathbb{N}$ find the conditional PMF of Y.

(b) Let X be a continuous random variable with the pdf

$$f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

Find the transformation Y=g(X) such that the pdf of Y will be

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

[Hint: Question 1(a) might be of use here !]

3. Suppose X and Y are independent Gaussian random variables with zero mean and variance σ^2. Show that $\frac{X}{Y}$ is cauchy.

4. (a) Particles are subject to collisions that cause them to split into two parts with each part a fraction of the parent. Suppose that this fraction is uniformly distributed between 0 and 1. Following a single particle through several splittings we obtain a fraction of the original particle $Z_n = X_1 X_2 ... X_n$ where each X_j is uniformly distributed between 0 and 1. Show that the density for the random variable Z_n is,

$$f_n(z) = \frac{1}{(n-1)!}(-log(z))^{n-1}$$

   (b) Suppose X and Y are independent exponential random variables with same parameter λ. Derive the pdf of the random variable $Z = \frac{\min(X,Y)}{\max(X,Y)}$.

5. A random variable Y has the pdf $f_Y(y) = Ky^{-(b+1)}, y \geq 2$ (and zero otherwise), where $b > 0$. This random variable is obtained as the monotonically increasing transformation $Y = g(X)$ of the random variable X with pdf $e^{-x}, x \geq 0$.

   (a) Determine K in terms of b.

   (b) Determine the transformation $g(.)$ in terms of b.

6. (a) Two particles start from the same point on a two-dimensional plane, and move with speed V each, such that the angle between them is uniformly distributed in $[0, 2\pi]$. Find the distribution of the magnitude of the relative velocity between the two particles.

   (b) A point is picked uniformly from inside a unit circle. What is the density of R, the distance of the point from the center?

7. Let X and Y be independent exponentially distributed random variables with parameter 1. Find the joint density of $U = X + Y$ and $V = \frac{X}{X+Y}$, and show that V is uniformly distributed.

References

[1] David Gamarnick and John Tsitsiklis, "Introduction to Probability", *MIT OCW*, , 2008.