EE5110 : Probability Foundations for Electrical Engineers **July-November 2015**

Lecture 15: Sums of Random Variables

Lecturer: Dr. Krishna Jagannathan *Scribes: R.Ravi Kiran*

15.1 Sum of Two Random Variables

In this section, we will study the distribution of the sum of two random variables. Before we discuss their distributions, we will first need to establish that the sum of two random variables is indeed a random variable.

Theorem 15.1 *Let X and Y be random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and define $Z(\omega) = X(\omega) + Y(\omega),\ \forall \omega \in \Omega$. Then, Z is a random variable.*

Proof: To prove that Z is a random variable, we need to show that $\{\omega \in \Omega : Z(\omega) > z\} \in \mathcal{F},\ \forall z \in \mathbb{R}$.

Now, $\forall z \in \mathbb{R}$, $Z(\omega) > z$ if and only if there exists a rational q such that $X(\omega) > q$ and $Y(\omega) > z - q$. This follows from the fact that the set of rationals is dense in $\mathbb{R}$. Thus,

$$\begin{aligned} \{\omega \in \Omega : Z(\omega) > z\} &= \bigcup_{q \in \mathbb{Q}} \{\omega \in \Omega : X(\omega) > q, Y(\omega) > z - q\} \\ &= \bigcup_{q \in \mathbb{Q}} \left(\{\omega \in \Omega : X(\omega) > q\} \cap \{\omega \in \Omega : Y(\omega) > z - q\}\right). \end{aligned} \tag{15.1}$$

We know that $\forall q \in \mathbb{Q}$, $\{\omega \in \Omega : X(\omega) > q\} \cap \{\omega \in \Omega : Y(\omega) > z - q\} \in \mathcal{F}$ because X and Y are random variables. Since the set of rationals is countable, we have a countable union of sets from $\mathcal{F}$, which should also be in $\mathcal{F}$ as it is a σ-algebra. Thus, $\{\omega \in \Omega : Z(\omega) > z\} \in \mathcal{F}$, proving that the sum, $Z = X + Y$ is a random variable. ∎

We will now start with random variables in the discrete domain. Assume that X and Y are discrete random variables with a known joint pmf $p_{X,Y}(\cdot)$. Let the random variable Z be defined as $Z = X + Y$. We will now characterize the pmf of Z, $p_Z(\cdot)$:

$$\begin{aligned} p_Z(z) &= \mathbb{P}(Z = z) \\ &= \sum_{x+y=z} p_{X,Y}(x, y) \\ &= \sum_x \mathbb{P}(X = x, Y = z - x) \\ &= \sum_x p_{X,Y}(x, z - x) \end{aligned} \tag{15.2}$$

In particular, if X and Y are independent, the pmf of Z simplifies to

$$p_Z(z) = \sum_x p_X(x) p_Y(z - x), \tag{15.3}$$

which is simply the discrete convolution of the two pmfs.

Let us now look at an example.

Example 15.2 *Let X and Y be independent, random variables with distributions given by $Pois(\lambda)$ and $Pois(\mu)$ respectively. Define $Z = X + Y$. Then, the pmf of Z, can be computed, by invoking (15.3) :*

$$\begin{aligned} p_Z(z) &= \sum_{x=0}^{z} \frac{e^{-\lambda}\lambda^x}{x!}\frac{e^{-\mu}\mu^{z-x}}{(z-x)!} \\ &= \frac{e^{-(\lambda+\mu)}}{z!}\sum_{x=0}^{z}\binom{z}{x}\lambda^x\mu^{z-x} \\ &= \frac{e^{-(\lambda+\mu)}(\lambda+\mu)^z}{z!} \end{aligned}$$

The above computation establishes that the sum of two independent Poisson distributed random variables, with mean values λ and μ, also has Poisson distribution of mean $\lambda + \mu$.

We can easily extend the same derivation to the case of a finite sum of independent Poisson distributed random variables.

Next, we consider the case of two jointly continuous random variables. Assume that X and Y are jointly continuous random variables, with joint pdf given by $f_{X,Y}(x, y)$. Let $Z = X + Y$. Then,

$$\begin{aligned} F_Z(z) &= \mathbb{P}(Z \leq z) \\ &= \mathbb{P}(X + Y \leq z) \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z-x} f_{X,Y}(x,y)dy\right)dx \qquad (15.4) \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z} f_{X,Y}(x,t-x)dt\right)dx \\ &= \int_{-\infty}^{z}\underbrace{\left(\int_{-\infty}^{\infty} f_{X,Y}(x,t-x)dx\right)}_{f_Z(t)}dt. \qquad (15.5) \end{aligned}$$

From (15.5), we can see that the pdf of Z is given by $f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z - x)dx$.

In the special case of X and Y being independent continuous random variables, we get

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx = f_X * f_Y, \qquad (15.6)$$

which is the convolution of the two marginal pdfs.

Example 15.3 *Assume that X_1 and X_2 are independent exponential random variables with parameters μ_1 and μ_2 respectively. Let $Z = X_1 + X_2$. Using (15.6) and the fact that the support for the exponential random variable is $\mathbb{R}^+ \cup \{0\}$, we get,*

$$\begin{aligned} f_Z(z) &= f_{X_1} * f_{X_2}, \\ &= \int_0^z \mu_1 e^{-\mu_1 x}\mu_2 e^{-\mu_2(z-x)}dx, \\ &= \mu_1\mu_2 e^{-\mu_2 z}\int_0^z e^{(\mu_2-\mu_1)x}dx. \end{aligned}$$

We can see from the above integral that

$$f_Z(z) = \begin{cases} \frac{\mu_1\mu_2}{\mu_2-\mu_1}\left(e^{-\mu_1 z} - e^{-\mu_2 z}\right) & \textit{if } \mu_1 \neq \mu_2, \\ \mu^2 z e^{-\mu z} & \mu_1 = \mu_2 = \mu. \end{cases}$$

In fact, the process can be extended to the case of a sum of a finite number n of random variables of distribution exp(μ), and we can observe that the pdf of the sum, Z_n, is given by Erlang (n, μ), i.e,

$$f_{Z_n}(z) = \frac{\mu^n z^{n-1} e^{-\mu z}}{(n-1)!}. \tag{15.7}$$

The above example describes the process of computing the pdf of a sum of continuous random variables.

The methods described above can be easily extended to deal with finite sums of random variables too.

15.2 Sum of a random number of random variables

In this section, we consider a sum of independent random variables, where the number of terms in the summation is itself random. Let N be a positive integer valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with known pmf $\mathbb{P}(N = n)$. Let $X_1, X_2, ...,$ be independent random variables on the same probability space, $(\Omega, \mathcal{F}, \mathbb{P})$, with distributions, $F_{X_1}(.), F_{X_2}(.), ...,$ respectively. Further, we will assume that N is independent of $\{X_i, i \geq 1\}$.

Define, $S_N = \sum\limits_{i=1}^{N} X_i$. That is, $S_N(\omega) = \sum\limits_{i=1}^{N(\omega)} X_i(\omega), \forall \omega \in \Omega$. The cdf of S_N can be computed as follows :

$$\begin{aligned} F_{S_N}(x) &= \mathbb{P}(S_N \leq x), \\ &= \sum_{k=1}^{\infty} \mathbb{P}(S_N \leq x | N = k)\mathbb{P}(N = k), \\ &= \sum_{k=1}^{\infty} \mathbb{P}(S_k \leq x)\mathbb{P}(N = k), \end{aligned} \tag{15.8}$$

where (15.8) follows from the independence of N and the X_is.

In the above expression, we know how to compute $\mathbb{P}(S_k \leq x)$ from the previous section. Thus we have essentially computed the distribution of the random sum of random variables under the specified independence assumptions.

The following example is quite instructive.

Example 15.4 *Geometric Sum of Exponentials :*
Let $X_i, \forall i \geq 1$ be independent random variables with distribution exp(μ). Let N be a positive integer valued random variable of geometric distribution with parameter p.

Define $S_N = \sum\limits_{i=1}^{N} X_i$. We will now determine the pdf of S_N.

We know that $\mathbb{P}(N = k) = (1-p)^{k-1}p, \forall k \geq 1$. Further we observed earlier (15.7) that the sum of k exponential distributions of mean $\frac{1}{\mu}$, $S_k = \sum\limits_{i=1}^{k} X_i$, is a k^{th} order Erlang distribution. Thus, using this and

(15.8), we get,

$$
\begin{aligned}
F_S(x) &= \mathbb{P}(S_N \leq x), \\
&= \sum_{k=1}^{\infty} \mathbb{P}(N = k) F_{S_k}(x), \\
&= \sum_{k=1}^{\infty} \left(p(1-p)^{k-1}\right)\left(1 - \sum_{n=0}^{k-1} \frac{1}{n!} e^{-\mu x} (\mu x)^n\right), \\
&= \sum_{k=1}^{\infty} p(1-p)^{k-1} - \sum_{k=1}^{\infty} p(1-p)^{k-1} e^{-\mu x} \left(\sum_{n=0}^{k-1} \frac{1}{n!} (\mu x)^n\right), \\
&= 1 - e^{-\mu x} \sum_{n=0}^{\infty} \frac{(\mu x)^n}{n!} \frac{p}{1-p} \sum_{k=n+1}^{\infty} (1-p)^k, \\
&= 1 - e^{-\mu x} \sum_{n=0}^{\infty} \frac{(\mu x(1-p))^n}{n!}, \\
&= 1 - e^{-\mu x} e^{\mu(1-p)x}, \\
&= 1 - e^{-(p\mu)x}.
\end{aligned}
$$

The above derivation establishes that the geometric sum of exponentials has an exponential distribution with parameter $\mu' = p\mu$.

Consider a radioactive source emitting α particles where the time between two successive emissions is exponentially distributed with parameter λ. Whenever there is an emission, the detector detects it with probability p and misses it with probability $1-p$ independent of other detections. So it can be easily seen that the time between two successive detections is indeed a geometric sum of i.i.d exponential random variables which itself is an exponential random variable with parameter $p\lambda$ as seen in the above example.

The above study gives a detailed account of the random sum of random variables under the strict independence constraints earlier assumed. It is however possible to envision a scenario where the random number N is dependent on the observations, X_i themselves.

For instance let us assume that a gambler plays a game repeatedly and is rewarded or penalized in each round. Say the gambler stops only when he is "satisfied" (or "broke") with the overall outcome of the game. Let X_i be the amount he gains (or loses) in round i of the game. In this scenario, analysing the overall sum earned by the gambler at the end of his game is complicated by the dependence of N on the outcomes. This scenario motivates the theory of *stopping rules*, which shall be covered in a more advanced course (EE6150).

15.3 Exercise:

1. Let X_1 and X_2 be independent random variables with distributions $\mathcal{N}(0, \sigma_1^2)$ and $\mathcal{N}(0, \sigma_2^2)$ respectively. Show that the distribution of $X_1 + X_2$ is $\mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$.

2. Consider two independent and identically distributed discrete random variables X and Y. Assume that their common PMF, denoted by $p(z)$, is symmetric around zero, i.e., $p(z) = p(-z), \forall z$. Show that the PMF of $X + Y$ is also symmetric around zero and is largest at zero.

3. Suppose X and Y are independent random variables with $Z = X + Y$ such that $f_X(x) = ce^{-cx}$, $x \geq 0$ and $f_Z(z) = c^2 z e^{-cz}$, $z \geq 0$. Compute $f_Y(y)$.

4. Let X_1 and X_2 be the number of calls arriving at a switching centre from two different localities at a given instant of time. X_1 and X_2 are well modelled as independent Poisson random variables with parameters λ_1 and λ_2 respectively.
   (a) Find the PMF of the total number of calls arriving at the switching centre.
   (b) Find the conditional PMF of X_1 given the total number of calls arriving at the switching centre is n.
5. The random variables X, Y and Z are independent and uniformly distributed between zero and one. Find the PDF of $X + Y + Z$.
6. Construct an example to show that the sum of a random number of independent normal random variables is not normal.