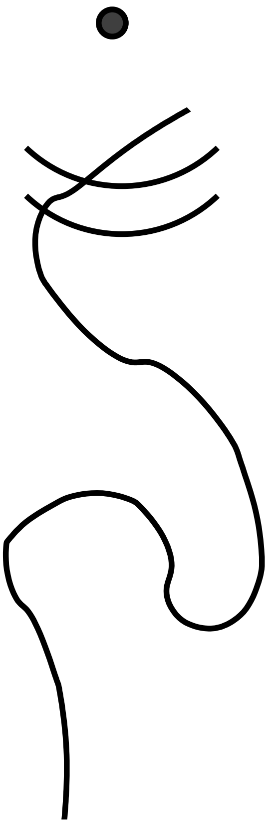
Work done to bring a charge from infinity

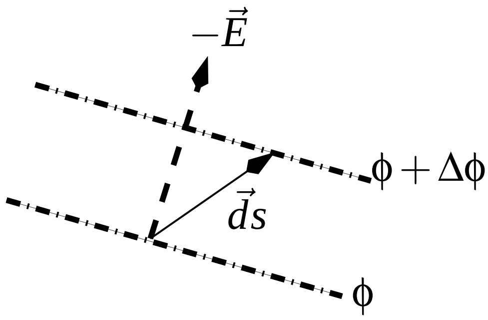
6th January 2007

Consider a charge Q at the origin. The field due to that charge is

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$

Another charge q is brought along an arbitrary path to a point $\vec{r}$.





The work done to bring the charge to its final point is given by

$$\int_{\infty}^{\vec{r}} \left(-q\vec{E}\right)\cdot \vec{ds}' = \left(-\frac{qQ}{4\pi\varepsilon_0}\right)\int_{\infty}^{\vec{r}} \frac{\hat{r'}\cdot\vec{ds}'}{r'^2} = \left(\frac{qQ}{4\pi\varepsilon_0}\right)\int_{r}^{\infty}\frac{dr'}{r'^2}$$

The last integral is really over $r(\vec{s})$. However, the integrand is a perfect derivitive, and the integral can be performed directly to yield

$$W = \left(\frac{qQ}{4\pi\varepsilon_0}\right)\frac{1}{r}$$

Thus the result is not a function of the path that got us there.

If you look at the derivation, any force that had the form $\vec{F} = f(r)\hat{r}$ would have yielded the same independence of path. Such forces are called central forces and are important for this reason. From a vector calculus point of view, they are force fields that have zero curl (try it out). Thus, given any two points P and Q, we can apply Stoke's Theorem to a closed loop consisting of an arbitrary path from P to Q and another arbitrary path from Q to P.

$$\int_P^Q \vec{F}\cdot d\vec{S}_1 + \int_Q^P \vec{F}\cdot d\vec{S}_2 = \oint \vec{F}\cdot d\vec{S} = \int\int \nabla\times\vec{F}\cdot d\vec{S} = 0$$

which means that the integral is path independent:

$$\int_P^Q \vec{F}\cdot d\vec{S}_1 = \int_P^Q \vec{F}\cdot d\vec{S}_2$$

Thus, it is the central nature of Coloumb's Law that yields $\nabla\times\vec{E} = 0$ and permits us to derive $\vec{E}$ from a scalar field ϕ.