The Wave Equation: Initial Value Example

17th March 2007

We start with the wave equation in 3 dimensions:

$$\nabla\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}=-\omega^2\mu\varepsilon\vec{E}$$

which has the solution

$$\vec{E}(\vec{r},t)=\iiint\left[\vec{A}(\vec{k})e^{j\left(kct-\vec{k}\cdot\vec{r}\right)}+\vec{B}(\vec{k})e^{j\left(-kct-\vec{k}\cdot\vec{r}\right)}\right]d^3k \tag{1}$$

Suppose we are given that at $t=0$,

$$\begin{aligned}\vec{E} &= 2\sin(k_0z)\,\hat{x}+3\cos(k_1y)\,\hat{z}\\ \eta\vec{H} &= \cos(k_1y)\,\hat{x}-\sin(k_2x)\,\hat{y}\end{aligned}$$

How do we find the fields at later time? At $t=0$ we have

$$\vec{E}=\iiint\left[\vec{A}(\vec{k})e^{j\left(-\vec{k}\cdot\vec{r}\right)}+\vec{B}(\vec{k})e^{j\left(-\vec{k}\cdot\vec{r}\right)}\right]d^3k$$

and we have from Faraday's law that

$$\vec{H}=-\frac{\nabla\times\vec{E}}{\pm j\omega\mu}$$

$$\vec{H}=\iiint\left[\frac{\vec{k}\times\vec{A}(\vec{k})}{\omega\mu}e^{j\left(-\vec{k}\cdot\vec{r}\right)}-\frac{\vec{k}\times\vec{B}(\vec{k})}{\omega\mu}e^{j\left(-\vec{k}\cdot\vec{r}\right)}\right]d^3k$$

where the minus sign comes from the time derivitive. We can also write this as

$$\eta\vec{H}=\iiint\left[\hat{k}\times\vec{A}e^{j\left(-\vec{k}\cdot\vec{r}\right)}-\hat{k}\times\vec{B}e^{j\left(-\vec{k}\cdot\vec{r}\right)}\right]d^3k$$

If $\vec{A}$ and $\vec{B}$ had constant directions, clearly, we can write from Eq. 1

$$\vec{E}(\vec{r},t)=\vec{f}(\vec{k}\cdot\vec{r}-kct)+\vec{g}(\vec{k}\cdot\vec{r}+kct)$$

Then, the magnetic field is given by

$$\vec{H}(\vec{r},t)=\frac{1}{\eta}\left[\hat{k}\times\vec{f}(\vec{k}\cdot\vec{r}-kct)-\hat{k}\times\vec{g}(\vec{k}\cdot\vec{r}+kct)\right]$$

Then,

$$\vec{E}\times\vec{H}=\frac{1}{\eta}\left(\vec{f}+\vec{g}\right)\times\left(\hat{k}\times\left(\vec{f}-\vec{g}\right)\right)=\frac{1}{\eta}\left[\vec{f}\times\left(\hat{k}\times\vec{f}\right)-\vec{g}\times\left(\hat{k}\times\vec{g}\right)\right]$$

where we have ignored the cross terms assuming them to have zero mean. For a uniform medium, the fields are perpendicular to $\vec{k}$, and the result is

$$\begin{aligned}
\vec{f}\times\left(\vec{k}\times\vec{f}\right) &= \varepsilon_{ijl}\hat{x}_i f_j\left(\varepsilon_{lmn}k_m f_n\right)\\
&= \left(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}\right)\hat{x}_i f_j k_m f_n\\
&= \vec{k}|f|^2-\vec{f}\left(\vec{f}\cdot\vec{k}\right)\\
&= \vec{k}|f|^2
\end{aligned}$$

So,

$$\vec{E}\times\vec{H}=\frac{\hat{k}}{\eta}\left(|f|^2-|g|^2\right)$$

as expected. The net flux of power along $\vec{k}$ is the difference in the power flux in the forward going wave and the backward going wave.

Now consider the initial conditions. There are only three k values present, namely $k_0\hat{z}$, $k_1\hat{y}$ and $k_2\hat{x}$. The directions are got from the multiplying coordinate, since it has to come out of $\vec{k}\cdot\vec{r}$. Since the equation is linear, we can break the problem into three problems, one in each k value.

$k_0\hat{z}$: $\vec{E}=2\sin(k_0z)\hat{x}$, $\vec{H}=0$ ($\vec{H}$ must be along $\hat{z}\times\hat{x}=\hat{y}$)

$k_1\hat{y}$: $\vec{E}=3\cos(k_1y)\hat{z}$, $\eta\vec{H}=\cos(k_1y)\hat{x}$ (The directions of $\vec{E}$ and $\vec{H}$ are consistent with each other)

$k_2\hat{x}$: $\vec{E}=0$, $\eta\vec{H}=-\sin(k_2x)\hat{y}$ ($\vec{E}$ must be along $\hat{z}$, so that $\hat{x}\times\hat{z}=-\hat{y}$)

We now solve these problems in turn.

1. For $k_0\hat{z}$: Since $\vec{H}$ is zero at $t=0$, we require $A-B=0$, i.e., $A(k_0)=B(k_0)$ and $A(-k_0)=B(-k_0)$. By inspection we can write

   $$\vec{E}(\vec{r},t)=\left(\sin(k_0z-\omega t)+\sin(k_0z+\omega t)\right)\hat{x}$$

   Then, the magnetic field is given by

   $$\eta\vec{H}(\vec{r},t)=\left(\sin(k_0z-\omega t)-\sin(k_0z+\omega t)\right)\hat{y}$$

2. For $k_2\hat{x}$: $\eta\vec{H}=-\sin(k_2x)\hat{y}$. $\vec{E}$ must be along $\hat{z}$, and is zero at $t=0$. Hence, $\vec{A}(k_2)=-\vec{B}(k_2)$. Again, by inspection, we can write

   $$\eta\vec{H}(\vec{r},t)=\left(-\sin(k_2x-\omega t)-\sin(k_2x+\omega t)\right)\frac{\hat{y}}{2}$$

   with

   $$\vec{E}(\vec{r},t)=\left(\sin(k_2x-\omega t)-\sin(k_2x+\omega t)\right)\frac{\hat{z}}{2}$$

   where I have used $\hat{z}\times-\hat{y}=\hat{x}$ to obtain the direction of $\vec{E}$.

3. For $k_1\hat{y}$: We have $\hat{z}\times\hat{x}=\hat{y}$, so the directions of $\vec{E}$ and $\vec{H}$ are consistent with each other. At $t=0$,

   $$\begin{aligned}
   \vec{E} &= 3\cos(k_1y)\hat{z}\\
   \vec{H} &= \cos(k_1y)\hat{x}
   \end{aligned}$$

Thus, at k_1,

$$\begin{aligned} A+B &= 3 \\ A-B &= 1 \end{aligned}$$

which yields $A(k_1)=2$, $B(k_1)=1$. Thus

$$\begin{aligned} \vec{E}(\vec{r},t) &= [2\cos(k_1y-\omega t)+\cos(k_1y+\omega t)]\,\hat{z} \\ \vec{H}(\vec{r},t) &= [2\cos(k_1y-\omega t)-\cos(k_1y+\omega t)]\,\hat{x} \end{aligned}$$

4. Suppose that in part 3, $\vec{H}(\vec{r},0)=\cos(k_1y)\hat{z}$. What then? If the wave is to propagate along $\vec{y}$, the E_z requires a H_x and a H_z requires a $-E_x$. Thus the solution then have been

$$\begin{aligned} \vec{E}(\vec{r},t) &= \frac{3}{2}[\cos(k_1y-\omega t)+\cos(k_1y+\omega t)]\,\hat{z} \\ &\quad -\frac{1}{2}[\cos(k_1y-\omega t)-\cos(k_1y+\omega t)]\,\hat{x} \\ \eta\vec{H}(\vec{r},t) &= \frac{3}{2}[\cos(k_1y-\omega t)-\cos(k_1y+\omega t)]\,\hat{x} \\ &\quad +\frac{1}{2}[\cos(k_1y-\omega t)+\cos(k_1y+\omega t)]\,\hat{z} \end{aligned}$$

Thus the missing components of the fields would have appeared once $t>0$.

5. Suppose that in part 3, $\vec{H}(\vec{r},0)=\sin(k_1y)\hat{x}$. Then, we would have had

$$\begin{aligned} f(z)+g(z) &= 3\cos(k_1y) \\ f(z)-g(z) &= \sin(k_1y) \end{aligned}$$

Clearly the solution is

$$\begin{aligned} f(z) &= \frac{3}{2}\cos(k_1y)+\frac{1}{2}\sin(k_1y) \\ g(z) &= \frac{3}{2}\cos(k_1y)-\frac{1}{2}\sin(k_1y) \end{aligned}$$

i.e.,

$$\begin{aligned} \vec{E}(\vec{r},t) &= \frac{3}{2}[\cos(k_1y-\omega t)+\cos(k_1y+\omega t)]\,\hat{z} \\ &\quad +\frac{1}{2}[\sin(k_1y-\omega t)-\sin(k_1y+\omega t)]\,\hat{z} \\ \vec{H}(\vec{r},t) &= \frac{3}{2}[\cos(k_1y-\omega t)-\cos(k_1y+\omega t)]\,\hat{x} \\ &\quad +\frac{1}{2}[\sin(k_1y-\omega t)+\sin(k_1y+\omega t)]\,\hat{x} \end{aligned}$$