# Vector Potential from Magnetic Field

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We start with the constitutive equation that relates force to field

$$\vec{F}(\vec{r}) = I_1 d\vec{l}_1 \times \vec{B}(\vec{r})$$

where

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{j}(r') \times \frac{\vec{R}_{12}}{R_{12}^3} dV' \tag{1}$$

Here $\vec{R}_{12} = \vec{r} - \vec{r}'$ is the position vector of the place where we are measuring field relative to where we have placed the current element $\vec{j}(r')dV'$. This relationship has been observed and we have motivated it from Coulomb's law and relativity.

The first thing to note is that $\vec{R}_{12}/R_{12}^3$ is the same factor that appears in Coulomb's law, and our previous work tells us that

$$\frac{\vec{R}_{12}}{R_{12}^3} = -\nabla_r \frac{1}{R_{12}} \tag{2}$$

where the subscript "r" in $\nabla_r$ indicates that the gradient is obtained by moving the measuring instrument, i.e., by varying $\vec{r}$. A quick calculation also tells us

$$\nabla_r \frac{1}{R_{12}} = -\nabla_{r'} \frac{1}{R_{12}} \tag{3}$$

where the second gradient is obtained by varying the source location. Remember that $R_{12}$ is a scalar function of *six* coordinates, namely $x, y, z, x', y', z'$.

We now put all this to work. The magnetic field can be written using Eq. 1 and 2 to get

$$\vec{B}(\vec{r}) = -\frac{\mu_0}{4\pi}\int \vec{j}(r') \times \nabla_r \frac{1}{R_{12}} dV' \tag{4}$$

We need to manipulate the cross product. For this

$$\begin{aligned}
\vec{j}(r') \times \nabla_r f &= \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ j_x(\vec{r}') & j_y(\vec{r}') & j_z(\vec{r}') \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
&= -\det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ j_x(\vec{r}') & j_y(\vec{r}') & j_z(\vec{r}') \end{vmatrix} \\
&= -\det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(\vec{r})j_x(\vec{r}') & f(\vec{r})j_y(\vec{r}') & f(\vec{r})j_z(\vec{r}') \end{vmatrix} \\
&= -\nabla \times \left(f(\vec{r})\vec{j}(\vec{r}')\right)
\end{aligned}$$

**In class, I forgot to interchange the rows as above. That is why I landed up with a '-' sign problem. Correct this in your notes.** Thus Eq. 4 becomes

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \nabla_r \times \frac{\vec{j}(\vec{r}')}{R_{12}(\vec{r},\vec{r}')} dV'$$

where $1/R_{12}$ was $f$ in the above derivation. Since the curl acts on $\vec{r}$ and the integral is over $\vec{r}'$, we can pull the curl out of the integral to get the result

$$\vec{B}(\vec{r}) = \nabla \times \vec{A}(\vec{r}) \tag{5}$$

where

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{r}')}{R_{12}(\vec{r},\vec{r}')} dV' \tag{6}$$

Note that Eq. 3 was not needed for this part. It will be very necessary later on.