Computing Vector Identities

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Vector identities are the biggest headaches of Electromagnetics. Bizarre identities appear and we are expected to feel happy about their appearance. Fortunately for those who aren't that fond of these manipulations, there are a couple of tricks that eliminate memorization of identities forever. I myself never remember any identity if I can help it. I can always derive what I need.

Einstein Notation

A vector is usually denoted through its components. For example

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

Since such expressions appear very often, a special notation has been invented. First we replace x, y and z by x_1, x_2 and x_3. Then the vector becomes

$$\vec{A} = \sum_{i=1}^{3} A_i\hat{x}_i \equiv A_i\hat{x}_i$$

The rule is that *if indices repeat in a product, they imply summation.* Now, a dot product of two vectors $\vec{A}$ and $\vec{B}$ is simply

$$A_iB_i$$

What about the sum of two vectors?

$$(A_i + B_i)\,\hat{x}_i$$

What about the ∇ operator?

$$\nabla = \frac{\partial}{\partial x_i}\hat{x}_i$$

Since vector calculus involves a lot of derivitives, and these are with respect to the coordinates, we shorten the notation $\partial/\partial x_i$ to ∂_i:

$$\nabla = \partial_i\hat{x}_i$$

So far so good. Let us write down Electrostatics:

$$\begin{aligned}
F_i &= qE_i && \text{Force Equation} \\
E_i &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho R_{12,i}}{R_{12}^3}dV' && \text{Coulomb's Law} \\
E_i &= -\partial_i\phi && \text{Electrostatic Potential} \\
\partial_i E_i &= \frac{\rho}{\varepsilon_0} && \text{Gauss' Law}
\end{aligned}$$

Let us do a little vector identity.

$$\begin{aligned}\nabla\cdot(fA) &= \partial_i(fA_i)\\ &= f\partial_i A_i + A_i\partial_i f\\ &= f\nabla\cdot\vec{A}+\vec{A}\cdot\nabla f\end{aligned}$$

It is that easy. Well, almost. The fly in the ointment is the curl.

Handling the Curl Operator

We know that (see "Cross Products and the Curl" writeup on the site).

$$\nabla\times\vec{A} = \det\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial_x & \partial_y & \partial_z\\ A_x & A_y & A_z\end{vmatrix}$$

How can we use the Einstein notation to write this out? As we discussed in the *Cross Products and the Curl* writeup, the determinant appears naturally in the curl and in dV because it is the natural way to count up volume. The determinant of a matrix is

- Linear in each row (and column) of the matrix.
- It takes a matrix and gives a number.
- It is invariant under cyclic rotations.
- It changes sign under exchanges.
- If any subspace of the matrix is linearly dependent (i.e., more than k of the vectors lie in a k-dimensional subspace), the determinant (i.e., the volume) goes to zero.

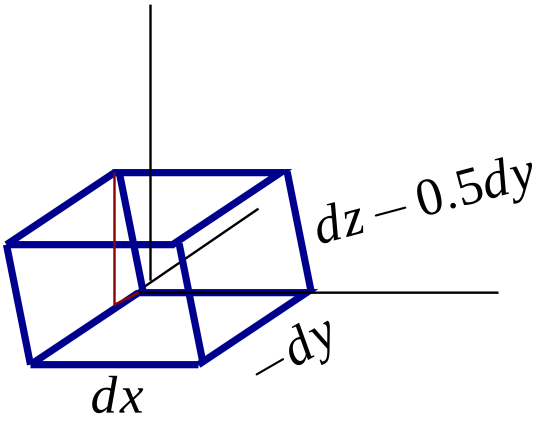
There is some other good stuff about the determinant, but what it adds up to is that the determinant is a very special function that takes matrices and gives you numbers. We can put down these properties algebraically as follows:

$$\vec{A}\times\vec{B} = \varepsilon_{ijk}\hat{x}_i A_j B_k$$

where ε_{ijk} is a *tensor* that has the following properties:

$$\varepsilon_{ijk} = \begin{cases} 1 & i,j,k \text{ cyclic}\\ -1 & i,j,k \text{ anti-cyclic}\\ 0 & \text{any two of } i,j,k \text{ equal}\end{cases}$$

The first term says that terms that are rotationally similar to xyz contribute to positive volume. The second term says that terms that are rotationally symmetric to xzy contribute to negative volume and terms that correspond to linearly dependent vectors is zero. For example, suppose I want to know the volume due to $u = x$, $v = -y$ and $w = z - 0.5y$. (du, dv, dw) form the sides of a parallelopiped



The volume of this parallelopiped is -$dxdydz$ since the area of the base is $dxdy$ and the height is dz. I can write the volume as

$$\varepsilon_{123}dx(-dy)dz+\varepsilon_{122}dx(-dy)(-0.5dy) = -dxdydz$$

Let us write out the cross product:

$$\begin{aligned}\left(\vec{A}\times\vec{B}\right)_x\hat{x} &= \varepsilon_{123}\hat{x}A_yB_z-\varepsilon_{132}\hat{x}A_zB_y\\ \left(\vec{A}\times\vec{B}\right)_y\hat{y} &= \varepsilon_{231}\hat{y}A_zB_x-\varepsilon_{213}\hat{y}A_xB_z\\ \left(\vec{A}\times\vec{B}\right)_z\hat{z} &= \varepsilon_{312}\hat{z}A_xB_y-\varepsilon_{321}\hat{z}A_yB_x\end{aligned}$$

Cyclic means $1,2,3$ or $2,3,1$ or $3,1,2$. Anti-cyclic means $1,3,2$ or $2,1,3$ or $3,2,1$. Anti-cyclic terms carry a negative sign as required by determinants.

But what have we achieved by writing things this way? Let us create a vector identity:

$$\begin{aligned}\nabla\times\nabla f &= \varepsilon_{ijk}\hat{x}_i\partial_j\left(\nabla f\right)\\ &= \varepsilon_{ijk}\hat{x}_i\partial_j\left(\partial_k f\right)\\ &= \left(\varepsilon_{ijk}\hat{x}_i\partial_j\partial_k\right)f\end{aligned}$$

But second partial derivitives commute, which means that we can rewrite the sum as

$$\nabla\times\nabla f = \sum_{i=1}^{3}\varepsilon_{ijk}\left(\partial_j\partial_k-\partial_k\partial_j\right)f = 0$$

where j and k are cyclic successors of i (if $i=2$, then $j=3, k=1$). Hence the determinant vanishes on account of identical rows. This is the proof that a vector field derived from a gradient has no curl.

$$\nabla\cdot\nabla\times\vec{A} = \partial_i\varepsilon_{ijk}\partial_jA_k = \varepsilon_{kij}\left(\partial_i\partial_j\right)A_k = 0$$

holds for the same reason. Again,

$$\begin{aligned}\vec{A}\times\vec{B} &= \varepsilon_{ijk}\hat{x}_iA_jB_k\\ &= -\varepsilon_{ikj}\hat{x}_iB_kA_j\\ &= -\vec{B}\times\vec{A}\end{aligned}$$

where we have used the fact that transposition of rows flips the sign (cyclic indices ijk became anti-cyclic).

Let us try another:

$$\begin{aligned}\nabla \times \left(f\vec{A}\right) &= \varepsilon_{ijk}\hat{x}_i\partial_j\left(fA_k\right) \\ &= f\varepsilon_{ijk}\hat{x}_i\left(\partial_j A_k\right) + \varepsilon_{ijk}\hat{x}_i\left(\partial_j f\right)A_k \\ &= f\nabla\times\vec{A} + \nabla f \times \vec{A}\end{aligned}$$

Still simple.

The final complication is what to do when we have two curls. For that we need to know how these ε tensors simplify when they multiply each other.

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \varepsilon_{ijk}\hat{x}_iA_j\left(\varepsilon_{klm}B_lC_m\right) = \varepsilon_{kij}\varepsilon_{klm}\hat{x}_iA_jB_lC_m$$

where the quantity in brackets is the k^{th} component of $\vec{B}\times\vec{C}$. We have a nasty expression. We have to simplify the product of tensors. For that note that both k,i,j and k,l,m must both be different for the product to be non-zero. So, if $k=1$, i and j can only be $2,3$ or $3,2$. And l and m can only be $2,3$ or $3,2$. So either $i=l$, $j=m$, or $i=m$, $j=l$. If $i=l$ and $j=m$, then both ijk and klm are cyclic or both are anti-cyclic. Either way, the product is $+1$. If $i=m$ and $j=l$, one of the terms is cyclic and the other is anti-symmetric. So the product is -1.

$$\varepsilon_{kij}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

The answer is now easy:

$$\begin{aligned}\vec{A}\times\left(\vec{B}\times\vec{C}\right) &= \left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)\hat{x}_iA_jB_lC_m \\ &= \hat{x}_iB_iA_jC_j - \hat{x}_iC_iA_jB_j \\ &= \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)\end{aligned}$$

The most famous example of this identity is the wave equation:

$$\begin{aligned}\nabla\times\left(\nabla\times\vec{E}\right) &= \left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)\hat{x}_i\partial_j\partial_lE_m \\ &= \nabla\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E}\end{aligned}$$

These rules are sufficient to solve the most complex of vector identities that you will face in Electromagnetics. For instance

$$\begin{aligned}\nabla\cdot\left(\vec{E}\times\vec{H}\right) &= \partial_i\left(\varepsilon_{ijk}E_jH_k\right) \\ &= \varepsilon_{ijk}\left(\partial_iE_j\right)H_k + \varepsilon_{ijk}\left(\partial_iH_k\right)E_j \\ &= \varepsilon_{kij}\left(\partial_iE_j\right)H_k + \varepsilon_{jki}\left(\partial_iH_k\right)E_j \\ &= \vec{H}\cdot\nabla\times\vec{E} - \vec{E}\cdot\nabla\times\vec{H}\end{aligned}$$

Additional Problems

Try your hand at the following:

1. $\vec{A} \times \left(\nabla \times \vec{B}\right)$
2. $\nabla \left(\vec{A} \cdot \vec{B}\right)$

   Hint: Use result of previous problem to express the result in terms of curl.
3. $\nabla \cdot (\nabla\phi \times \nabla\psi)$