Uniqueness of Solutions to Poisson's Equation

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The question that this theorem tries to answer is:

> Suppose we follow some algorithm and find a solution to Poisson's Equation. Is it the only answer?

The answer is fairly easy to obtain. But we need some vector calculus preliminaries. The fundamental identity we need is to determine the divergence of a product of a scalar and a vector (specifically the gradient of ϕ).

$$\begin{aligned}
\nabla\cdot(f\vec{g}) &= \frac{\partial}{\partial x}(fg_x)+\frac{\partial}{\partial y}(fg_y)+\frac{\partial}{\partial z}(fg_z) \qquad (1)\\
&= \frac{\partial f}{\partial x}g_x+\frac{\partial f}{\partial y}g_y+\frac{\partial f}{\partial z}g_z\\
&\quad +f\frac{\partial}{\partial x}(g_x)+f\frac{\partial}{\partial y}(g_y)+f\frac{\partial}{\partial z}(g_z)\\
&= \nabla f\cdot\vec{g}+f\nabla\cdot\vec{g}
\end{aligned}$$

We start with the basic equation (I am anticipating the displacement vector here, which you already know about from your physics course)

$$\nabla\cdot\vec{D}=\rho \qquad (2)$$

Expressing in terms of the potential this yields

$$\nabla\cdot(\varepsilon\nabla\phi)=-\rho \qquad (3)$$

where $\varepsilon(r)$ is the spatially varying dielectric constant. Consider two solutions ϕ_1 and ϕ_2, both of which satisfy Eq. 3 and the boundary conditions. Let $u=\phi_1-\phi_2$ be the difference of these solutions. Then,

$$\nabla\cdot(\varepsilon\nabla u)=0$$

and u goes to zero on the boundary (we assume that the boundary has potential specified on it).

We now apply Eq. 1 to $f=u$ and $g=\varepsilon\nabla u$ to get

$$\nabla\cdot(\varepsilon u\nabla u)=u\nabla\cdot(\varepsilon\nabla u)+\nabla u\cdot\varepsilon\nabla u=\varepsilon\left|\nabla u\right|^2$$

Now apply the divergence theorem to this result:

$$\int\varepsilon\left|\nabla u\right|^2 dV=\int\nabla\cdot(\varepsilon u\nabla u)=\oint\varepsilon u\nabla u\cdot\vec{dS}$$

But $u = \phi_1 - \phi_2$ is either zero or its derivitive is zero at every point on the surface, since both ϕ_1 and ϕ_2 satisfied the same boundary conditions. Thus, the last integral is zero and we obtain

$$\int \varepsilon \, |\nabla u|^2 \, dV = 0 \tag{4}$$

Now ε is a real positive function for static fields (ε can be complex when we write phasor equations), and $|\nabla u|^2$ is positive definite in u. u is the difference of two potentials, each of which is double differentiable since it is a solution of Poisson's Equation. Hence the only way Eq. 4 can be satisfied is if u is identically zero everywhere. If u were nonzero anywhere, it must have a nonzero slope somewhere in the volume, since $u = 0$ at the boundary. That region of nonzero slope would make the integral in Eq. 4 nonzero as well.

There is an exception to this result: if the boundaries were specified entirely in terms of $\nabla\phi \cdot \hat{dS} \equiv \partial\phi / \partial n$. But this is a trivial exception since ϕ has no meaning for this problem and only its derivitives are meaningful quantities.