The Time Dependent Equations of E&M

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The static laws of Electricity and Magnetism give us the following equations:

$$\begin{array}{rl} \vec{E} & \vec{B} \\ \nabla \times \vec{E} = 0 & \nabla \cdot \vec{B} = 0 \\ \nabla \cdot \vec{D} = \rho & \nabla \times \vec{H} = \vec{j} \end{array} \tag{1}$$

These equations are adequate for time-independent phenomena. However, they fail for time dependent situations. There are two kinds of failures that occur:

1. Failure of conservation laws.

2. Failure of translation symmetries.

Both of these failures make this system of equations unacceptable as descriptions of physical phenomena.

1 Charge Conservation

Charge is one of the fundamental invariants known to physics. It cannot be created or destroyed, and it is invariant under all inertial transformations. Thus, in all coordinate systems, we expect the following to hold:

$$\oint_S \vec{j} \cdot d\vec{S} = -\frac{d}{dt} \int_V \rho d^3 r = -\frac{dQ}{dt}$$

The differential form of this equation is

$$\nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t} \tag{2}$$

If we take the divergence of Ampere's law, we obtain

$$\nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot \vec{j} \neq -\frac{\partial \rho}{\partial t}$$

To make Ampere's law consistent with charge conservation, we use Poisson's equation and relate ρ to $\vec{D}$:

$$-\frac{\partial \rho}{\partial t} = -\frac{\partial \nabla \cdot \vec{D}}{\partial t} = \nabla \cdot \left(-\frac{\partial \vec{D}}{\partial t} \right)$$

Thus we require
$$\nabla \cdot \left(\nabla \times \vec{H} - \vec{j}\right) = \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$
The revised Ampere's law therefore becomes
$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

2 Translation Invariance

A charge q that is moving with a velocity $\vec{v}$ experiences a force given by
$$q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$
This force is measured in the Lab Frame. If we measure the same force in a frame moving at constant velocity $\vec{v}$, the force should be the same. However, in that frame, the relative velocity of the charge is zero. So the force is given by
$$q\vec{E}'$$
where $\vec{E}'$ is the Electric Field in the moving Frame. This yields a transformation for the Electric Field:
$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$$
Now this new Electric Field is uniquely determined by its divergence and its curl. Let us look at what these are:
$$\begin{aligned}
\nabla' \cdot \vec{E}' &= \nabla \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right) \\
&= \frac{\rho}{\varepsilon_0} + \nabla \cdot \left(\vec{v} \times \vec{B}\right) \\
&= \frac{\rho}{\varepsilon_0} + v_j \varepsilon_{ijk} \partial_i B_k \\
&= \frac{\rho}{\varepsilon_0} - \vec{v} \cdot \nabla \times \vec{B} \\
&= \frac{\rho}{\varepsilon_0} - \mu_0 \vec{v} \cdot \vec{j} \\
&= \frac{\rho}{\varepsilon_0} - \frac{1}{\varepsilon_0 c^2} v^2 \rho \\
&= \frac{\rho}{\varepsilon_0}\left(1 - \frac{v^2}{c^2}\right)
\end{aligned}$$
where we have used $\nabla' \cdot = \nabla \cdot$ (derived below) and $\vec{j} = \rho\vec{v}$. All this is saying is that in the rest frame, lengths are longest, which means that the distance between charges has increased, i.e., charge density has reduced. This is just a relativistic correction to ρ. **Note:** Actually the expected relativistic correction should be $1 - v^2 \big/ 2c^2$. Possibly an additional relativistic correction came from the transformation of ∇.

The curl equation for the Electric Field says that

$$\nabla \times \vec{E} = 0$$

in all frames. Applying this to $\vec{E}'$ yields

$$\nabla \times \vec{E}' = 0 = \nabla \times \vec{E} + \nabla \times \left(\vec{v} \times \vec{B}\right)$$

i.e.,

$$\nabla \times \left(\vec{v} \times \vec{B}\right) = 0$$

This is obviously false, since $\vec{v}$ could be in anything we choose. Using $\nabla \cdot \vec{B} = 0$, the primed equation becomes

$$\nabla \times \vec{E}' = -(\vec{v} \cdot \nabla)\vec{B}$$

What can we add to the curl equation to make it look the same in all inertial frames? The clue is given by the right side of the primed equation. That term is the *convective derivative* of the magnetic field. It appears when we look at the rate of change in time in a moving frame. Consider the following change of coordinates

$$x = x' + vt', \quad t = t'$$

The partial derivatives transform as

$$\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial t} + \underline{v} \cdot \nabla$$

and

$$\frac{\partial}{\partial x'} = \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}$$

We therefore try

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

In the primed frame, this equation becomes

$$\begin{aligned} \nabla \times \vec{E}' &= \nabla \times \vec{E} - (\vec{v} \cdot \nabla)\vec{B} \\ &= -\partial_t \vec{B} - (\vec{v} \cdot \nabla)\vec{B} \\ &= -\partial_{t'} \vec{B} \end{aligned}$$

This new equation is therefore translation invariant.

Note: The mistake I made in class was to transform ∂_t in terms of $\partial_{t'}$ and $\partial_{x'}$. It is the reverse transformation I require.

We have only used "Galilean invariance" in the derivation above. What that means is that Special Relativity was not invoked, i.e., that these transformations are only acceptible when $v \ll c$. However, Faraday's Law is actually true very generally, and is an integral part of any "covarient" description of Electromagnetics.

3 Doesn't $\vec{B}$ Transform as well?

To derive Faraday's Law, we transformed the Electric Field but kept the Magnetic Field constant. Is that correct?

A simple thought experiment will prove that it is not correct. Consider a line charge along the z-axis with ρ Coul/metre that is at rest in the Lab frame. Now go to a frame that is moving with a velocity $v_0\hat{z}$ with respect to the Lab. The line charge is moving in this frame, which means that it represents both a charge and a current. Thus, in the Lab frame, there is only an Electric Field, but in the moving Frame, there is both an Electric and a Magnetic Field.

What saves us are the magnitudes of these effects. Any such term is important only when the velocity approaches the speed of light, since

$$\frac{\left|\vec{v}\times\vec{B}\right|}{E} \simeq \frac{\mu_0 v j/4\pi r^2}{\rho/4\pi\varepsilon_0 r^2} \simeq \frac{v^2}{c^2}$$

when we use $\vec{j} = \rho\vec{v}$. Thus, for non-relativistic systems, it is acceptible to consider that the Magnetic Field does not transform while the Electric Field does.

4 Consequences of Special Relativity

From the viewpoint of relativity, there is only one field. In some cases, it manifests Electric Field-like properties. In other situations, it looks like a Magnetic Field. Even the curl and the divergence collapse into a single tensor operation. The equations of E&M become:

$$\partial_\alpha J^\alpha = 0 \quad \text{charge conservation}$$

$$\Box A^\alpha = \mu J^\alpha \quad \text{wave equation}$$

$$\partial_\alpha A^\alpha = 0 \quad \text{Lorentz gauge condition}$$

$$\partial_\alpha F^{\alpha\beta} = \mu J^\beta \quad \text{source Maxwell equations}$$

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \quad \text{homogeneous Maxwell equations}$$

In these equations, $F^{\alpha\beta}$ is the Field Strength Tensor, given by

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & -B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & B_y & B_x & 0 \end{pmatrix},$$

∂_α is the space-time derivitive and $\Box$ is the wave operator.