Finding the field due to a solenoid

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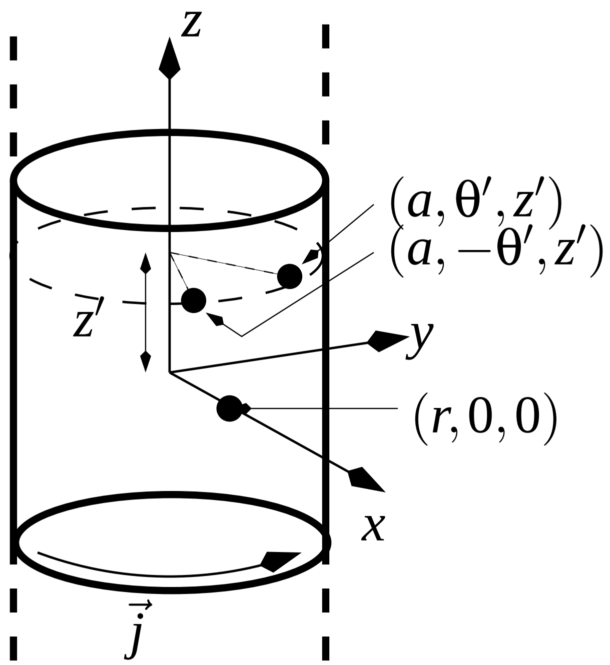
Consider an ideal solenoid of radius a, with n turns per metre, carrying current I_0. We wish to compute the magnetic field $\vec{B}$ due to these currents.



As discussed in the class, we have a problem that is symmetric in θ and z. Thus,

$$\vec{B}(\vec{r}) = B_r(r)\hat{r} + B_\theta(r)\hat{\theta} + B_z(r)\hat{z} \tag{1}$$

From the fact that magnetic field has zero divergence ... why zero divergence?

$$\begin{aligned}\nabla\cdot\vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi}\int \nabla\cdot\left(\vec{j}(\vec{r}')\times\nabla\frac{1}{R_{12}}\right)dV' \\ &= -\frac{\mu_0}{4\pi}\int \nabla\cdot\left(\nabla\times\frac{\vec{j}(\vec{r}')}{R_{12}}\right)dV' \\ &= 0\end{aligned}$$

since the divergence of a curl is always zero, from stokes theorem. Note that our ε notation for curl (see the vector identities writeup) helps us derive step 2 from step 1:

$$\begin{aligned}\vec{j}(\vec{r}')\times\nabla\frac{1}{R_{12}} &= \varepsilon_{klm}\hat{x}_k j_l \partial_m \frac{1}{R_{12}} \\ &= -\varepsilon_{kml}\hat{x}_k\partial_m\left(\frac{j_l}{R_{12}}\right) \\ &= -\nabla\times\left(\frac{\vec{j}(\vec{r}')}{R_{12}}\right)\end{aligned}$$

It is worth noting that relativity said force due to magnetic fields were given by $\left(1/4\pi\varepsilon_0 c^2\right) I_1\vec{d}l_1 \times \left(I_2\vec{d}l_2\times\vec{R}_{12}/R_{12}^3\right)$, and this same form now implies that the divergence of $\vec{B}$ is zero. This property of magnetic fields is built into the very process by which they are derived from Coulomb's law.

So going back to Eq. 1, we obtain

$$\nabla\cdot\vec{B}=\frac{1}{r}\partial_r\left(rB_r\right)+\frac{1}{r}\partial_\theta B_\theta+\partial_z B_z=\frac{1}{r}\partial_r\left(rB_r\right)=0$$

Hence $B_r=B_0/r$, which is not allowed if the domain includes $r=0$. i.e., $B_r\equiv 0$. So,

$$\vec{B}(\vec{r})=B_\theta\hat{\theta}+B_z\hat{z}$$

This magnetic field is obtained from

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\int\vec{j}(\vec{r}')\times\nabla\frac{1}{R_{12}}dV'$$

where $\vec{j}=I_0 n\hat{\theta}'\delta(r'-a)$. To make progress we go to the Vector Potential

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{j}(\vec{r}')}{R_{12}}dV'$$

Since the problem is symmetric in θ and z, we assume $\theta=0$ and $z=0$. $\vec{j}(\vec{r}')$ is along $\hat{\theta}'$, i.e., along $-\hat{x}\sin\theta'+\hat{y}\cos\theta'$. Then,

$$\begin{aligned}\vec{A}(r) &= \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}dz'\int_{-\pi}^{\pi}ad\theta'\frac{I_0 n\left(-\sin(\theta')\hat{x}+\cos(\theta')\hat{y}\right)}{\sqrt{\left(x-a\cos\theta'\right)^2+a^2\sin^2\theta'+z'^2}}\\ &= \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}dz'\int_{-\pi}^{\pi}ad\theta'\frac{I_0 n\cos(\theta')\hat{y}}{\sqrt{\left(x-a\cos\theta'\right)^2+a^2\sin^2\theta'+z'^2}}\\ &= \frac{\mu_0}{4\pi}I_0 n\hat{y}\int_{-\infty}^{\infty}dz'\int_{-\pi}^{\pi}ad\theta'\frac{\cos(\theta')}{\sqrt{\left(x-a\cos\theta'\right)^2+a^2\sin^2\theta'+z'^2}}\end{aligned}$$

For $\theta=0$, $\hat{y}$ is the same as $\hat{\theta}$, since $\hat{r}$ is along $\hat{x}$. Thus, this shows us that $\vec{A}(r)$ is along $\hat{\theta}$ which is what we wanted to get out of this exercise.

We are now almost done. If $\vec{A}$ is along θ and depends only on r, its curl

$$\frac{1}{r}\det\begin{vmatrix}\hat{r} & r\hat{\theta} & \hat{z}\\ \partial_r & \partial_\theta & \partial_z\\ 0 & rA_\theta & 0\end{vmatrix}=\frac{1}{r}\partial_r\left(rA_\theta\right)\hat{z}$$

$\vec{B}$ is purely along $\hat{z}$ and depends only on r. We could use the above formulae to calculate it, but there is now an easier way. We use Stoke's theorem on a loop in the $r-z$ plane, with unit length along z, and its two sides at $r=0$ and at r.

$$\begin{aligned}\oint\vec{B}\cdot\vec{dl} &= B_z(0)-B_z(r)\\ &= \mu_0 I_{\mathrm{encl}}\end{aligned}$$

Thus, we immediately conclude (assuming $B_z\rightarrow 0$ as $r\rightarrow\infty$)

$$B_z(r)=\begin{cases}\mu_0 nI_0 & r<a\\ 0 & r>a\end{cases}$$

Finite Solenoid

What happens when the solenoid is not infinitely long?

∂_z is no longer zero, and so, B_r can now be present:

$$\nabla\cdot\vec{B}=\frac{1}{r}\partial_r\left(rB_r\right)+\partial_z B_z=0 \tag{2}$$

The arguments we used for $\vec{A}$ do not change; the only difference is that the limits in z are now finite. So we have

$$\vec{A}(\vec{r})=A_\theta(r,z)\hat{\theta}$$

Taking the curl yields

$$\vec{B}(\vec{r})=-\hat{r}\partial_z A_\theta+\hat{z}\frac{1}{r}\partial_r\left(rA_\theta\right)$$

The divergence should automatically go to zero, as it does:

$$\frac{1}{r}\partial_r\left(-r\partial_z A_\theta\right)+\partial_z\left(\frac{1}{r}\partial_r\left(rA_\theta\right)\right)=0$$

since second partial derivitives commute.

As a special case, consider $r \ll a$. The expression for A_θ becomes, for small x

$$\begin{aligned}
A_\theta &= \frac{\mu_0}{4\pi} I_0 n\hat{y}\int_{-L/2}^{L/2} d\zeta \int_{-\pi}^{\pi} a d\theta' \frac{\cos(\theta')}{\sqrt{(x-a\cos\theta')^2+a^2\sin^2\theta'+\zeta^2}} \\
&\simeq \frac{\mu_0}{4\pi} I_0 n\hat{y}\int_{-L/2-z}^{L/2-z} d\zeta \int_{-\pi}^{\pi} a d\theta' \frac{\cos(\theta')}{\sqrt{a^2+\zeta^2-2ax\cos\theta'}} \\
&\simeq \frac{\mu_0}{4\pi} I_0 n\hat{y}\int_{-L/2-z}^{L/2-z} d\zeta \int_{-\pi}^{\pi} a d\theta' \frac{\cos(\theta')}{\sqrt{a^2+\zeta^2}}\left(1+\frac{ax\cos(\theta')}{a^2+\zeta^2}\right) \\
&= \frac{\mu_0}{4} I_0 n\hat{y} a^2 x \int_{-L/2-z}^{L/2-z} d\zeta \frac{1}{(a^2+\zeta^2)^{3/2}}
\end{aligned}$$

The integral is known in closed form (look up your favourite table of integrals. It is in all of them):

$$\int_\alpha^\beta dz \frac{a^2}{(a^2+z^2)^{3/2}} = \frac{\beta}{\sqrt{\beta^2+a^2}}-\frac{\alpha}{\sqrt{\alpha^2+a^2}}$$

Applying the formula above, we obtain A_θ and B_z for the infinite case:

$$\begin{aligned}
A_\theta &= \frac{\mu_0}{2} I_0 n r \\
B_z &= \frac{1}{r}\partial_r\left(rA_\theta\right) = \frac{\mu_0}{2}\frac{I_0 n}{r}\partial_r\left(r^2\right) = \mu_0 n I_0
\end{aligned}$$

For the finite case, this becomes

$$\begin{aligned}
B_z &= \mu_0 n I_0 \int_{-L/2-z}^{L/2-z} d\zeta \frac{a^2}{2(a^2+\zeta^2)^{3/2}} \\
&= \mu_0 n I_0 \left[\frac{L/2-z}{\sqrt{\left(L/2-z\right)^2+a^2}}-\frac{-L/2-z}{\sqrt{\left(-L/2-z\right)^2+a^2}}\right]
\end{aligned}$$

It is worth noting that the fringing fields of a solenoid *do not fall off exponentially*! Remember the case of the capacitor. There the fringing fields fell off exponentially. The difference between the two cases is that for the capacitor, we specified *voltage* rather than $\sigma(x)$ on the plates. Here, we have specified the precise currents. When we do that, we do not allow for "self-consistent" currents, and the fringing fields can penetrate a lot further.