Inductance of a solenoid

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1 Basic Equations

We start with the curl equations

$$\begin{aligned} \nabla \times \vec{E} &= -j\omega\vec{B} \\ \nabla \times \vec{H} &= \vec{J} = \sigma\vec{E} \end{aligned}$$

Substituting one into the other, and using our previous arguments to discard $\nabla \cdot \vec{E}$, we obtain

$$\nabla^2\vec{E} = j\omega\mu\sigma\vec{E}$$

Now the current is in the $\hat{\theta}$ direction, which means that $\vec{E}$ is also in that direction. However, $\vec{E}$ is independent of θ and z. So we obtain

$$\frac{1}{r}\partial_r\left(r\partial_r E_\theta\right)\hat{\theta} + \frac{E_\theta}{r^2}\partial_\theta^2\hat{\theta} = j\omega\mu\sigma E_\theta\hat{\theta}$$

From our course in mechanics we learned that

$$\partial_\theta^2\hat{\theta} = -\hat{\theta}$$

So, we obtain,

$$\frac{1}{r}\partial_r\left(r\partial_r E_\theta\right)\hat{\theta} + \frac{E_\theta}{r^2}\partial_\theta^2\hat{\theta} = j\omega\mu\sigma E_\theta\hat{\theta}$$

Dropping the $\hat{\theta}$ we get

$$\frac{1}{r}\partial_r\left(r\partial_r E_\theta\right) - \frac{1}{r^2}E_\theta - j\omega\mu\sigma E_\theta = 0$$

Multiplying through by r^2 we obtain

$$r^2E_\theta'' + rE_\theta' + \left(-1 - j\omega\mu\sigma r^2\right)E_\theta = 0$$

We transform to $\rho = r\left(1+j\right)/\delta$ to get

$$\rho^2E_\theta'' + \rho E_\theta' - \left(1+\rho^2\right)E_\theta = 0$$

The solutions are modified Bessel functions

$$E_\theta = AI_1\left(\rho\right) + BK_1\left(\rho\right)$$

Assuming the inner and outer radii are a and b, and $a, b-a \gg \delta$, we can use the asymptotic approximations for these functions, i.e.,

$$\begin{aligned} I_1(\rho) &\approx e^{\rho} \Big/ \sqrt{2\pi\rho} \\ K_1(\rho) &\approx e^{-\rho} \sqrt{\pi/2\rho} \end{aligned}$$

The field can then be written

$$E_\theta(r) = \frac{1}{\sqrt{r}} \left(A e^{r(1+j)\big/\delta} + B e^{-r(1+j)\big/\delta} \right)$$

The magnetic field is then given by

$$B_z = \frac{1}{r} \partial_r \left(r \left[A I_1(\rho) + B K_1(\rho) \right] \right) = A I_0(\rho) + B K_0(\rho)$$

Now the magnetic field must go to zero at $r = b$ and its value at $r = a$ is $B_z = \mu n I$. Hence,

$$B_z \approx \mu n I \sqrt{\frac{a}{r}} \frac{\sinh\left(\frac{(b-r)(1+j)}{\delta}\right)}{\sinh\left(\frac{(b-a)(1+j)}{\delta}\right)}$$

where we have used the asymptotic expressions for the Bessel functions. For $b - a \gg \delta$, this further simplifies to

$$B_z \approx \mu n I \sqrt{\frac{a}{r}} e^{-(r-a)\big/\delta} e^{-j(r-a)\big/\delta}$$

As expected the magnitude decays on a skin depth.

2 The Electric Field

The Electric Field can be obtained from Faraday's Law. In the vacuum region inside the coils, the magnetic field is uniform and given by $\mu n I$. Hence,

$$\frac{1}{r} \partial_r (r E_\theta) = -j\omega\mu n I = \text{Constant}$$

Hence, we can solve to get

$$E_\theta = -\frac{j\omega\mu n I}{2} r, \qquad 0 < r < a$$

Beyond a, Faraday's law again gives us

$$\frac{1}{r} \partial_r (r E_\theta) \approx -j\omega\mu n I \sqrt{\frac{a}{r}} e^{-(r-a)\big/\delta} e^{-j(r-a)\big/\delta}$$

Solving this, and assuming that the exponential's variation is the dominant term,

$$E_\theta = j\omega\mu n I \sqrt{\frac{a}{r}} \frac{\delta}{1+j} \exp\left(-\frac{1+j}{\delta} (r-a) \right), \qquad r \gg a$$

Note that E_θ is not continuous at $r = a$, which needs discussion in a separate section.

3 Inductance

The current through the coil is I. And the magnetic field is proportional to this current in the vacuum regions since

$$B_z = \mu n I$$

The Electric field has been obtained above

$$E_\theta = j\omega\mu n I \sqrt{\frac{a}{r}} \frac{\delta}{1+j} \exp\left(-\frac{1+j}{\delta}(r-a)\right)$$

Thus, at $r = a$, $E_\theta = C(1+j)I$. This indicates that the voltage is only 45^o out of phase with the current, rather than 90^o. We are therefore faced with the problem of how to identify the connection between the internal Electric field and the externally applied voltage.

The answer is given in Ramo and Whinnery, section 4.11:

- We start with the expression for $\vec{E}$ in terms of potentials (derived in class):

$$\vec{E} = -\nabla\phi - \partial_t \vec{A}$$

- Now, there is a current that leads to a resistive loss in the circuit:

$$\vec{J} = \sigma \vec{E}$$

- However, this $\vec{E}$ is not the same as the field above. This Electric field is the sum of the applied Electric Field and any other induced Electric Fields:

$$\vec{E} = \vec{E}_{\text{applied}} + \vec{E}_{\text{induced}}$$

- The applied field is due to a battery. So our equation becomes

$$\vec{E}_{\text{applied}} + \left(-\nabla\phi - \partial_t \vec{A}\right) = \frac{\vec{J}}{\sigma}$$

- Applying the loop integral to this equation, and choosing a contour that hugs the inner wall of the solenoid (i.e., $r = a$) we obtain

$$\mathscr{E}_{\text{applied}} - \frac{d}{dt}\int\int \vec{B}\cdot\vec{dS} = \frac{\vec{J}}{\sigma}$$

  or, using the expressions above,

$$\mathscr{E}_{\text{applied}} = j\omega\mu\pi a^2 n I + 2\pi a n I\left(\frac{j\omega\mu\delta}{1+j}\right)$$

  where $r = a$ has been used in the skin depth equation. Taking terms common this yields:

$$\mathscr{E}_{\text{applied}} = (j\omega)(\mu n I)\left(\pi a^2\right)\left(1 + \frac{\delta}{a}(1-j)\right)$$

  This is where External and Internal inductance come from - for large conductivity or high frequency, external inductance dominates. But when $a \simeq \delta$, the internal inductance becomes important. The $-\delta/a$ term is a resistance, since the coefficient of I is

$$V = \left(\omega\mu n\pi a^2 \frac{\delta}{a}\right) I = \left(2\pi a\sqrt{\frac{\pi f \mu}{\sigma}}\right) I$$

  At higher frequencies and for lower conductivities, the resistance increases.