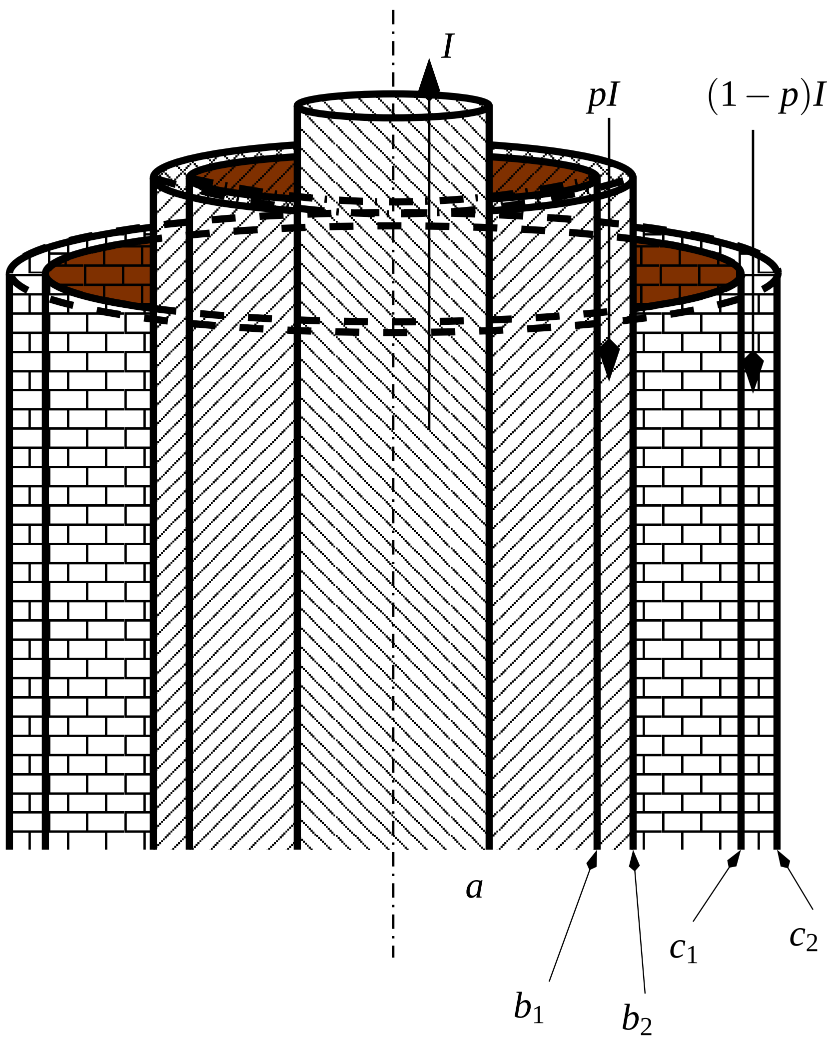
The Skin Effect in a set of three Concentric Cylinders

24th February 2007

Let us work out the skin effect in the following geometry



We have three coaxial cylinders. The inner one is hollow, and the middle and outer ones are hollow. All the solid portions have a conductivity σ and the gaps are air-filled.

A current I flows up through the central conductor and currents pI and $(1-p)I$ flow back down the outer conductors as shown. The problem is to determine the detailed current distributions.

The skin effect is the equation to be obeyed, and the same arguments used to discuss the arbitrary cylindrical wire apply here. Once again, I use i instead of j to represent $\sqrt{-1}$

$$\begin{aligned}\nabla\times\vec{E} &= -i\omega\vec{B}\\ \nabla\times\vec{H} &= \vec{j}\\ \vec{j} &= \sigma\vec{E}\\ \vec{B} &= \mu_0\vec{H}\end{aligned}$$

Hence, we obtain

$$-\nabla_\perp^2 E_z = -i\omega\mu_0\sigma E_z$$

In cylindrical coordinates, with the only coordinate of interest being r, this becomes

$$\frac{1}{r}\partial_r\left(r\partial_r E_z\right) - i\omega\mu_0\sigma E_z = 0$$

or

$$r^2\frac{d^2E_z}{dz^2} + r\frac{dE_z}{dr} - i\omega\mu_0\sigma r^2 E_z = 0$$

The solutions are in the form of Bessel functions,

$$E_z = AJ_0(\lambda r) + BY_0(\lambda r)$$

where $\lambda = \sqrt{\pi f\mu_0\sigma}\,(1-i) = (1-i)/\delta$, where δ is the skin depth. The current and the magnetic field are then given by

$$j_z = \sigma AJ_0(\lambda r) + \sigma BY_0(\lambda r)$$

and

$$-\partial_r\left(E_z\right) = -i\omega\mu_0 H_\theta$$

i.e.,

$$H_\theta = \frac{\lambda}{i\omega\mu_0}\left(AJ_0'(\lambda r) + BY_0'(\lambda r)\right)$$

where $J_0'(\lambda r) \equiv dJ_0(\lambda r)\big/d(\lambda r)$.

Asymptotically, for $\lambda r \gg 1$

$$J_0(\lambda r) \simeq \sqrt{\frac{2}{\pi\lambda r}}\cos\left(\lambda r - \pi/4\right)$$

In $r < a$,

$$\begin{aligned}E_z &= A_1 J_0(\lambda r)\\ j_z &= \sigma A_1 J_0(\lambda r)\\ H_\theta &= \frac{\lambda}{i\omega\mu_0}A_1 J_0'(\lambda r)\end{aligned}$$

since Y_0 blows up at $r = 0$. In $b_1 < r < b_2$

$$\begin{aligned}
E_z &= A_2 J_0(\lambda r) + B_2 Y_0(\lambda r) \\
j_z &= \sigma A_2 J_0(\lambda r) + \sigma B_2 Y_0(\lambda r) \\
H_\theta &= \frac{\lambda}{i\omega\mu_0}\left(A_2 J_0'(\lambda r) + B_2' Y_0(\lambda r)\right)
\end{aligned}$$

In $c_1 < r < c_2$,

$$\begin{aligned}
E_z &= A_3 J_0(\lambda r) + B_3 Y_0(\lambda r) \\
j_z &= \sigma A_3 J_0(\lambda r) + \sigma B_3 Y_0(\lambda r) \\
H_\theta &= \frac{\lambda}{i\omega\mu_0}\left(A_3 J_0'(\lambda r) + B_3' Y_0(\lambda r)\right)
\end{aligned}$$

Now in the airgaps, the symmetry immediately allows us to conclude:

$$\begin{aligned}
H_\theta &= \tfrac{I}{2\pi r} && a < r < b_1 \\
&= \tfrac{pI}{2\pi r} && b_2 < r < c_1 \\
&= 0 && r > c_2
\end{aligned}$$

Thus, we can use the continuity of H_θ to determine the unknown coefficients:

$$\begin{aligned}
\frac{\lambda}{i\omega\mu_0} J_0'(\lambda a) A_1 &= \frac{I}{2\pi a} \\
\frac{\lambda}{i\omega\mu_0}\left(A_2 J_0'(\lambda b_1) + B_2 Y_0'(\lambda b_1)\right) &= \frac{I}{2\pi b_1} \\
\frac{\lambda}{i\omega\mu_0}\left(A_2 J_0'(\lambda b_2) + B_2 Y_0'(\lambda b_2)\right) &= \frac{pI}{2\pi b_2} \\
\frac{\lambda}{i\omega\mu_0}\left(A_3 J_0'(\lambda c_1) + B_3 Y_0'(\lambda c_1)\right) &= \frac{pI}{2\pi c_1} \\
\frac{\lambda}{i\omega\mu_0}\left(A_3 J_0'(\lambda c_2) + B_3 Y_0'(\lambda c_2)\right) &= 0
\end{aligned}$$

or, defining $\alpha = i\omega\mu_0 I / 2\pi\lambda$, we have

$$\begin{aligned}
A_1 &= \frac{2\alpha}{J_0'(\lambda a)} \\
\begin{pmatrix} J_0'(\lambda b_1) & Y_0'(\lambda b_1) \\ J_0'(\lambda b_2) & Y_0'(\lambda b_2) \end{pmatrix}\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} &= \begin{pmatrix} \alpha/b_1 \\ p\alpha/b_2 \end{pmatrix} \\
\begin{pmatrix} J_0'(\lambda c_1) & Y_0'(\lambda c_1) \\ J_0'(\lambda c_2) & Y_0'(\lambda c_2) \end{pmatrix}\begin{pmatrix} A_3 \\ B_3 \end{pmatrix} &= \begin{pmatrix} p\alpha/c_1 \\ 0 \end{pmatrix}
\end{aligned}$$

The coefficients can be solved by inverting the matrix and noting that $J_0'(z) = -J_1(z)$.

$$\begin{aligned}
\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} &= \frac{-1}{J_1(\lambda b_1)Y_1(\lambda b_2) - J_1(\lambda b_2)Y_1(\lambda b_1)}\begin{pmatrix} Y_1(\lambda b_2) & -Y_1(\lambda b_1) \\ -J_1(\lambda b_2) & J_1(\lambda b_2) \end{pmatrix}\begin{pmatrix} \alpha/b_1 \\ p\alpha/b_2 \end{pmatrix} \\
\begin{pmatrix} A_3 \\ B_3 \end{pmatrix} &= \frac{-1}{J_1(\lambda c_1)Y_1(\lambda c_2) - J_1(\lambda c_2)Y_1(\lambda c_1)}\begin{pmatrix} Y_1(\lambda c_2) & -Y_1(\lambda c_1) \\ -J_1(\lambda c_2) & J_1(\lambda c_2) \end{pmatrix}\begin{pmatrix} p\alpha/c_1 \\ 0 \end{pmatrix} \qquad (1)
\end{aligned}$$

Clearly the answer scales with α which it should, since α contains I. We can easily solve for the coefficients and hence have the complete answer. The solution shape depends on a/δ, b_1/δ, b_2/δ, c_1/δ, c_2/δ and p. Using these are inputs, we can graph the answer.

The Code

First we define the parameters:

4a ⟨* 4a⟩≡

```
a=3;b1=4;b2=12;c1=13;c2=16;p=0.5;n=201;
```

Now define some utility functions. `dj` and `dy` compute $J_0'(\lambda z)$ and $Y_0'(\lambda z)$.

4b ⟨* 4a⟩+≡

```
deff("y=dj(z)","z1=z*(1+%i);y=-besselj(1,z1)");
deff("y=dy(z)","z1=z*(1+%i);y=-bessely(1,z1)");
```

Routine `jz` returns a vector of j_z values if a vector of r values are passed to it.

4c ⟨* 4a⟩+≡

```
deff("y=jz(A,B,r)","z1=r*(1+%i); ...
  y=A*besselj(0,z1);if B<>0, ...
  y=A*besselj(0,z1)+B*bessely(0,z1),end");
```

Routine `fac` computes the determinant and inverts it.

4d ⟨* 4a⟩+≡

```
deff("y=fac(e1,e2)", ...
  "y=1/(dj(b1)*dy(b2)-dj(b2)*dy(b1))");
```

Routines `jmax` and `jmin` are used to fix graph bounds. `jmin` has a check to limit the lower bound since we want semi-log plots.

4e ⟨* 4a⟩+≡

```
deff("y=jmax(j)","y=max(abs(j))");
deff("y=jmin(j)","y=floor(min(abs(j))); ...
  if y==0,y=1e-3*jmax(j),end");
```

Compute the coefficients now, using Eq. 1.

4f ⟨* 4a⟩+≡

```
A1=2/dj(a); // $A_1$ is directly calculated.
v=[dy(b2) -dy(b1); -dj(b2) dj(b1)] ...
  *fac(b1,b2)*[1/b1;p/b2];
A2=v(1);B2=v(2);
w=[dy(c2) -dy(c1);-dj(c2) dj(c1)] ...
  *fac(c1,c2)*[p/c1;0];
A3=w(1);B3=w(2);
```

Plot the graphs using subplots.

5 ⟨* 4a⟩+≡

```
xset("window",0);
xset("thickness",3);
subplot(3,1,1);
r=linspace(0,a,n);
j=jz(A1,0,r);
plot2d(r,abs(j),rect=[0 jmin(j) a jmax(j)], ...
  logflag="nl");
xset("thickness",0);
xgrid(5);
xtitle("Region 1");
subplot(3,1,2);
r=linspace(b1,b2,n);
j=jz(A2,B2,r);
xset("thickness",3);
plot2d(r,abs(j),rect=[b1 jmin(j) ...
  b2 jmax(j)],logflag="nl");
xset("thickness",0);
xgrid(5);
xtitle("Region 2");
subplot(3,1,3);
r=linspace(c1,c2,n);
j=jz(A3,B3,r);
xset("thickness",3);
plot2d(r,abs(j),rect=[c1 jmin(j) ..
  c2 jmax(j)],logflag="nl");
xset("thickness",0);
xgrid(5);
xtitle("Region 3");
```

Discussion

Existence of Solutions

The code assumes that the coefficient matrix is invertible. But is it? Can't we think of a particular σ or ω at which the oscillatory part of $J_0(\lambda r)$ goes to zero? Then we won't be able to match at all.

To answer this question we have to go back to the equation for j_z:

$$\frac{1}{r}\partial_r\left(r\partial_r j_z\right)+\lambda^2 j_z=0$$

The solutions are $AJ_0(\lambda z)+BY_0(\lambda z)$. We have boundary conditions on the derivitive at b_1 and b_2 (say). Sturm Louville theory says (which is why you need to have learned it during your Maths courses) that there is a complete set of eigenvalues and eigenfunctions that can build up any j_z. Since these functions are purely oscillatory along the real axis, the eigen values are all real, and so only for real λ is it possible to find solutions of

$$\det\begin{vmatrix} J_0'(\lambda b_1) & Y_0'(\lambda b_1) \\ J_0'(\lambda b_2) & Y_0'(\lambda b_2) \end{vmatrix}=0$$

since that is the eigenvalue equation. Hence, for complex λ as we have for skin effect, this determinant can never go to zero.

But why does our argument above fail? Doesn't the oscillatory part go to zero infinite number of times? The answer is that

$$J_0(\lambda r)=\mathrm{Ber}(\lambda r)+i\mathrm{Bei}(\lambda r)$$

and each of Ber and Bei go to zero nearly periodically. However, *they do not go to zero at the same location*! That is to say, the real part or the imaginary part may go to zero, but the complex function never goes to zero.

This is easier to understand in the cartesian case. Then,

$$B_y=A\sin(\lambda x)=A\sin(x/\delta)\cosh(x/\delta)+iA\cos(x/\delta)\sinh(x/\delta)$$

Clearly both cos and sin go to zero periodically. But never for the same x. If we now obtain the actual field in time, we get

$$\begin{aligned}
B_y(x,t) &= A\left(\sin(x/\delta)\cosh(x/\delta)\cos\omega t-\cos(x/\delta)\sinh(x/\delta)\sin\omega t\right)\\
&\simeq \frac{A}{2}e^{x/\delta}\left(\sin\left(x/\delta\right)\cos\omega t-\cos\left(x/\delta\right)\sin\omega t\right)\\
&= B_y(a)\frac{e^{-(a-x)/\delta}}{2}\sin\left(\frac{x}{\delta}-\omega t\right)
\end{aligned}$$

No zeros! Instead we have something new ... $\sin\left(x/\delta-\omega t\right)$, which is what we call a "travelling wave".

Nature of Solutions

Figure 1 shows typical solutions obtained from the code. The plots are semi-log plots and show that j_z decays exponentially as we move away from any of the walls.

One thing to note is that the current does not go to zero at c_2. Rather it is the derivitive of j_z that goes to zero there.

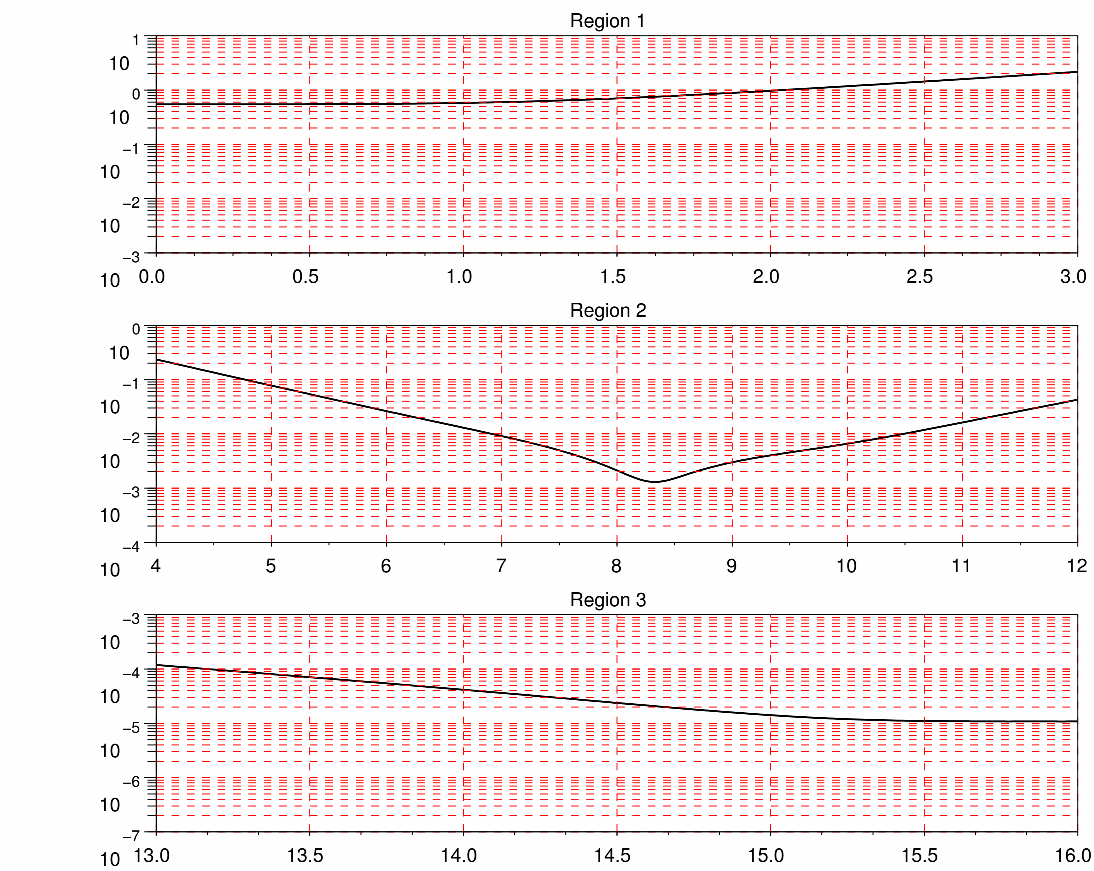


Figure 1: Solutions for the case $a = 3\delta$, $b_1 = 4\delta$, $b_2 = 12\delta$, $c_1 = 13\delta$, $c_2 = 16\delta$ and $p = 0.5$.

Another thing to note is that we cannot make c_2 any bigger. The problem is that the solution involves the difference of nearly equal quantities (like cosh and sinh. The computer loses numerical precision when trying to compute the answer.

To understand these results, we must recognise that for $|z| \gg 1$,

$$\begin{aligned}
J_0(z) &\simeq \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right) \\
Y_0(z) &\simeq \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4}\right) \\
J_1(z) &\simeq \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4}\right) \\
Y_1(z) &\simeq -\sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right)
\end{aligned}$$

Thus, for distances much larger than a skin depth, the Bessel functions reduce to trigonometric functions. Then, the middle region equation looks like (for $b_2 - b_1 \ll b_1 + b_2$)

$$\begin{pmatrix} \sin(\lambda b_1 - \frac{\pi}{4}) & -\cos(\lambda b_1 - \frac{\pi}{4}) \\ \sin(\lambda b_2 - \frac{\pi}{4}) & -\cos(\lambda b_2 - \frac{\pi}{4}) \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} \alpha/b_1 \\ p\alpha/b_2 \end{pmatrix}$$

Since $\lambda = \delta(1+i)$, we have nearly equal, exponentially large, coefficients. We can always replace the trigonometric terms by complex exponentials and only the real part contributes to magnitude. So the equation looks like so (with $C = -A_2 - B_2$ and $D = A_2 - B_2$):

$$\begin{pmatrix} e^{b_1(1-i)/\delta} & e^{-b_1(1-i)/\delta} \\ e^{b_2(1-i)/\delta} & e^{-b_2(1-i)/\delta} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} \simeq \begin{pmatrix} 2\alpha/b_1 \\ 2p\alpha/b_2 \end{pmatrix}$$

When $b_2 - b_1 \gg \delta$, the boundary condition at b_1 is primarily fixed by the decaying exponential, while the boundary condition at b_2 is fixed by the growing exponential:

$$\begin{aligned}
C &\simeq \left(\frac{p\alpha}{b_2}\right) e^{-b_2(1-i)/\delta} \\
D &\simeq \left(\frac{\alpha}{b_1}\right) e^{b_1(1-i)/\delta}
\end{aligned}$$

We can confirm by substituting back into the solution

$$\begin{aligned}
e^{b_1(1-i)/\delta} C + e^{-b_1(1-i)/\delta} D &= \left(\frac{\alpha}{b_1}\right) + \left(\frac{p\alpha}{b_2}\right) e^{-(b_2-b_1)(1-i)/\delta} \\
e^{b_2(1-i)/\delta} C + e^{-b_2(1-i)/\delta} D &= \left(\frac{\alpha}{b_1}\right) e^{-(b_2-b_1)(1-i)/\delta} + \left(\frac{p\alpha}{b_2}\right)
\end{aligned}$$

This is why we see a skin on either side of the middle shell, with almost no current in the middle.