The Skin Effect

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The skin effect is the most important low-frequency effect in Electromagnetics. The basic effect is discussed in the textbook, and you should read it carefully. What is discussed here is how the skin effect works on a cylindrical wire of arbitrary cross-section.

We start with the basic equations

$$
\begin{aligned}
\nabla\cdot\vec{B} &= 0\\
\nabla\times\vec{H} &= \vec{j}\\
\nabla\times\vec{E} &= -\partial_t\vec{B}\\
\vec{B} &= \mu\vec{H}\\
\vec{j} &= \sigma\left(\vec{E}+\frac{\vec{j}}{ne}\times\vec{B}\right)\\
\vec{E} &= -\nabla\phi+\vec{E}_{\text{ind}}
\end{aligned}
$$

Now, the applied Electric Field is along $\hat{z}$. The current must be along $\vec{z}$ for the reasons already discussed in connection with the D.C. case. That means $\vec{A}$ is along $\hat{z}$ and $\vec{B}$ has only $\vec{x}$ and $\vec{y}$ components.

Since $\vec{B}$ has zero divergence, it cannot have lines that spiral out, and its field lines must be closed loops. **Note:** In 3-D this result does not work, since lines do not have to close on themselves but can fill a surface. But since $\vec{B}$ is in the x-y plane, the field lines have to strictly close on themselves.

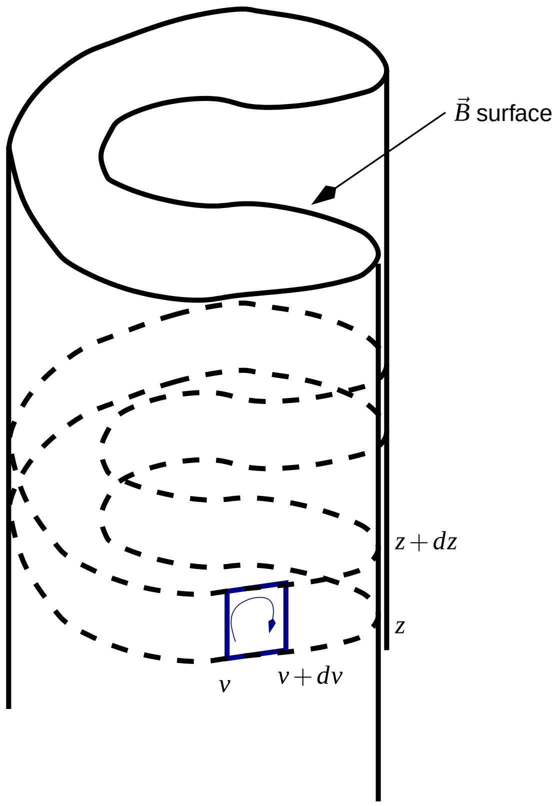
We now show that j_z is constant on a "magnetic surface" (the surface got by taking a magnetic loop and translating it along z. Consider

$$\nabla\times\vec{E} = -j\omega\vec{B}$$

But (I use i here for $\sqrt{-1}$ to avoid confusion with $\vec{j}$)

$$
\begin{aligned}
\nabla\times\vec{j} &= \sigma\nabla\times\left(\vec{E}+\frac{\vec{j}\times\vec{B}}{ne}\right)\\
&= -i\omega\sigma\vec{B}+\frac{\sigma}{ne}\nabla\times\left(\vec{j}\times\vec{B}\right)\\
&= -i\omega\sigma\vec{B}+\frac{\sigma}{ne}\left(\vec{j}\nabla\cdot\vec{B}-\vec{B}\nabla\cdot\vec{j}\right)\\
&= \left(-i\omega\sigma+\frac{\sigma}{ne}\partial_t\rho\right)\vec{B}
\end{aligned}
$$

Thus, the curl of $\vec{j}$ points along $\vec{B}$. Consider a stokes surface as shown in the figure.



Since $\vec{j}$ is only along $\hat{z}$, the loop integral says

$$\oint \vec{j}\cdot\vec{dl} = \left(j_z(u,v) - j_z(u,v+dv)\right)dz = \int \nabla\times\vec{j}\cdot\vec{dS} = 0$$

The loop integral is zero since the curl of $\vec{j}$ has no component normal to $\vec{B}$. Thus, $\vec{j}$ is constant on any magnetic surface. The skin effect equation can now be derived

$$\begin{aligned} \nabla\times\left(\nabla\times\vec{E}\right) &= -i\omega\mu\nabla\times\vec{B} \\ &= -i\omega\mu\vec{j} \end{aligned}$$

i.e.,

$$\nabla_\perp^2 j_z - i\omega\mu\sigma j_z = 0$$

For E_z, the equation becomes

$$\nabla_\perp^2 E_z - i\omega\mu\sigma E_z = 0$$

This equation does not tell us what E_z is going to do. However, suppose we defined our coordinate system as (u,v,z) as shown in the figure. Then, we know that both the curl of $\vec{E}$ and of $\vec{j}$ are along $\vec{B}$, and hence, both are constant on the magnetic surfaces. The equation then becomes of the form

$$\frac{1}{A(u)}\partial_u\left(A(u)\partial_u E_z\right) - i\omega\mu\sigma E_z = 0$$

where $A(u)$ is the area (per metre along z) of the magnetic surface at u. But this is the standard form of a Sturm-Louville equation, and the solutions are going to be a product of exponential-like and trigonometric-like functions with u scaled by $\delta = 1\big/\sqrt{\pi f \mu \sigma}$ (here I assume that u is, like r, an actual distance). It immediately means that even for irregular shaped wires, the current is to be found hugging the boundaries of the wire.