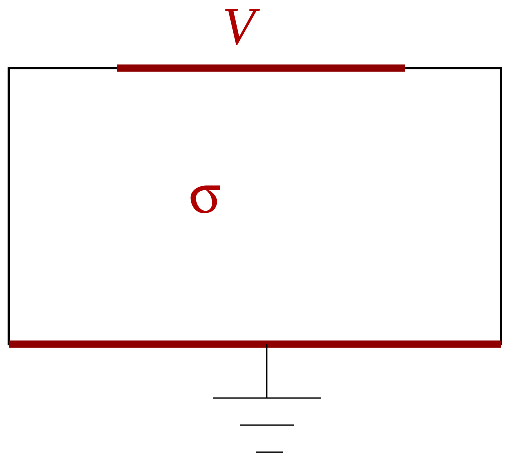
Resistance of a Device

21st January 2007

Consider the following device



The material has a conductivity σ and its top lead extends from $-a < x < a$, while the device extends from $-L < x < L$. The device is d metres in height along $\hat{y}$. The problem is to find the current that is injected.

The boundary conditions are

$$\begin{aligned} \phi &= 0, \quad y = 0 \\ \frac{\partial \phi}{\partial x} &= 0 \quad x = -L, L \\ \frac{\partial \phi}{\partial y} &= 0 \quad y = d, -a > x \text{ or } a < x \\ \phi &= V \quad y = d, -a < x < a \end{aligned}$$

This is a “mixed” boundary condition problem, since the top wall specifies ϕ along part of the wall and $\partial\phi/\partial y$ along the rest. This is a hard a problem to solve, since we do not have basis functions to handle it.

A Homogeneous Problem

Let us simplify the problem by saying that a constant current density j_0 is injected through the top lead. So the top wall has a boundary condition that is

$$\frac{\partial \phi}{\partial y} = 0 \qquad |x| > a$$

$$\frac{\partial\phi}{\partial y} \quad = -j_0/\sigma \quad |x| < a$$

We now construct the solution using separation of variables, assuming $\phi(x,y) = F(x)G(y)$. Along x, the solution is symmetric about $x = 0$ and we can write

$$F(x) = \cos\left(\frac{n\pi}{L}x\right)$$

Along y, the zero boundary condition at $y = 0$ immediately yields

$$G(y) = \sinh\left(\frac{n\pi}{L}y\right)$$

For the special case of $n = 0$, $G'' = 0$ which yields $G(y) = By$ as the solution to use. The full solution therefore becomes

$$\phi(x,y) = By + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi}{L}x\right) \frac{\sinh\left(\frac{n\pi}{L}y\right)}{\sinh\left(\frac{n\pi}{L}d\right)}$$

At $y = d$, we have the final boundary condition, namely

$$Bd + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{L}x\right) = \begin{cases} j_0 & |x| < a \\ 0 & |x| > a \end{cases}$$

This permits us to determine the unknown coefficients B and c_n. If we only want to find B, though we can just integrate both sides along x. The sum vanishes as all the coefficients have zero average. That yields

$$2BdL = 2j_0a$$

or $B = j_0a/L$. The voltage at $x = 0$ could be called the lead voltage, which is gvien by

$$V = Bd + \sum_{n=1}^{\infty} c_n = \frac{j_0a}{L}d + \frac{2j_0}{L}\sum_{n=1}^{\infty} \frac{L}{n\pi} \sin\left(\frac{n\pi}{L}a\right)$$

The resistance of the lead could be defined as the ratio of V to the total current. This yields

$$R = \frac{V}{2j_0a} = \frac{d}{2L} + \frac{1}{a\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{L}a\right)$$

The Mixed Boundary Problem

Suppose on the other hand, we insisted on a voltage condition at the lead. We use symmetry to solve only half the region, namely $0 < x < L$ and $0 < y < d$. This region is divided into two parts, namely Region I with $0 < x < a$ and $0 < y < d$ and Region II with $a < x < L$ and $0 < y < d$.

In Region I, the solution can be written as

$$\phi_I(x,y) = V\frac{y}{d} + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{d}y\right) \frac{\cosh\left(\frac{n\pi}{d}x\right)}{\cosh\left(\frac{n\pi}{d}a\right)}$$

In Region II, the solution becomes

$$\phi_{II}(x,y) = \sum_{n=0}^{\infty} d_n \sin\left(\frac{(n+0.5)\pi}{d}y\right) \frac{\cosh\left(\frac{(n+0.5)\pi}{d}(L-x)\right)}{\cosh\left(\frac{(n+0.5)\pi}{d}(L-a)\right)}$$

At $x = a$, both the potential and the normal component of current are continuous. Hence, we obtain two equations to connect c_n and d_n. Continuity of potential yields

$$V\frac{y}{d} + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{d}y\right) = \sum_{n=0}^{\infty} d_n \sin\left(\frac{(n+0.5)\pi}{d}y\right)$$

and continuity of current says

$$\begin{aligned} \sum_{n=1}^{\infty} c_n \frac{n\pi}{d} \sin\left(\frac{n\pi}{d}y\right) \tanh\left(\frac{n\pi}{d}a\right) \quad = \quad & \sum_{n=0}^{\infty} d_n \frac{(n+0.5)\pi}{d} \sin\left(\frac{(n+0.5)\pi}{d}y\right) \\ & \times \tanh\left(\frac{(n+0.5)\pi}{d}(L-a)\right) \end{aligned}$$

These two equations offer a system of equations with which to solve for c_n and d_n. However this is an infinite system of equations, which is no better than the equation we started with!

This is why mixed boundary condition problems are so hard. They are best solved numerically.