Deducing the Magnetic Field from Relativity

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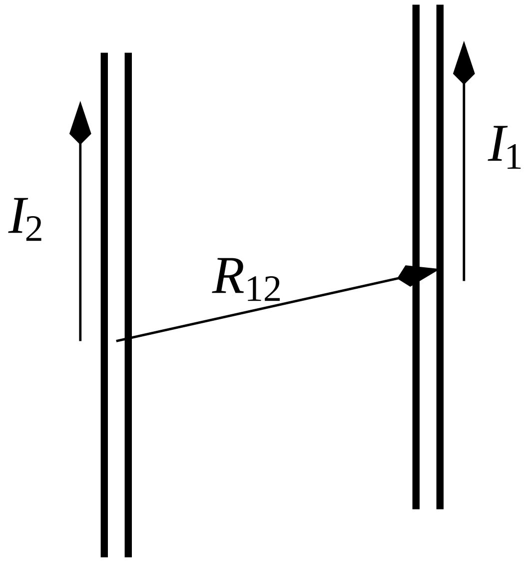
This is adapted from Robert Feynmann's lectures on physics.

Consider two wires that are infinitely long and carry currents I_1 and I_2. The wires are identical and parallel, and are at a distance R apart.



Let us calculate the force per metre of wire 2 on wire 1. The number of conduction electrons is assumed to be n per unit length of the wire, and are assumed to be one per atom. Since the wire is neutral, that is also the number of "ions", which are really the nucleii together with other shielding electrons. The current I_1 must come from the motion of electrons, i.e.,

$$I_1 = -ne\vec{v_1}$$

where $\vec{v_1}$ is the velocity of electrons in wire 1. Similarly,

$$I_2 = -ne\vec{v_2}$$

From the point of the ions in wire 1, there is a repulsive force from ions in wire 1 and in wire 2. The ions in wire 1 cannot do anything, since they are part of a rigid body and self forces add up to zero (a body cannot push itself). The ions in wire 2 exert a force equal to

$$\vec{F^+} = -\frac{1}{4\pi\varepsilon_0}(ne)^2\frac{\vec{R_{12}}}{R_{12}^3}$$

The electrons in wire 2 are moving with velocity $\vec{v_2}$. Relativistic contraction means that the density of electrons increases slightly. The attractive force becomes

$$\vec{F^-} = \frac{1}{4\pi\varepsilon_0}(ne)\left(\frac{ne}{\sqrt{1-v_2^2/c^2}}\right)\frac{\vec{R_{12}}}{R_{12}^3}$$

Thus, the total external force experienced by the ions is given by

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}(ne)^2\frac{\vec{R_{12}}}{R_{12}^3}\left(\frac{1}{\sqrt{1-v_2^2/c^2}}-1\right) \simeq \frac{1}{4\pi\varepsilon_0}\frac{(nev_2)^2}{2c^2}\frac{\vec{R_{12}}}{R_{12}^3}$$

The electrons in wire 1 are moving with velocity $\vec{v}_1$. To them, the ions in wire 2 move with velocity $-\vec{v}_1$ (so do ions in wire 1, but those can't do anything useful). The same arguments now say

$$\vec{F^+} = \frac{1}{4\pi\varepsilon_0}(ne)\left(\frac{ne}{\sqrt{1-v_1^2/c^2}}\right)\frac{\vec{R_{12}}}{R_{12}^3}$$

The electrons in wire 2 move with a velocity equal to $\vec{v}_2 - \vec{v}_1$ with respect to the electrons in wire 1. This veloctiy depends on the directions of the two currents. So

$$\vec{F^-} = -\frac{1}{4\pi\varepsilon_0}(ne)\left(\frac{ne}{\sqrt{1-|v_1-v_2|^2/c^2}}\right)\frac{\vec{R_{12}}}{R_{12}^3}$$

The total force on the electrons becomes

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}(ne)\left(\frac{ne}{\sqrt{1-v_1^2/c^2}} - \frac{ne}{\sqrt{1-|v_1-v_2|^2/c^2}}\right)\frac{\vec{R_{12}}}{R_{12}^3}$$

For small velocities, this becomes

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}(ne)^2\left(\frac{v_1^2}{2c^2} - \frac{(v_1-v_2)^2}{2c^2}\right)\frac{\vec{R_{12}}}{R_{12}^3} = -\frac{1}{4\pi\varepsilon_0}(ne)^2\left(\frac{v_2^2-2v_1v_2}{2c^2}\right)\frac{\vec{R_{12}}}{R_{12}^3}$$

Adding both forces, the total force on the wire becomes,

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}(ne)^2\frac{2v_1v_2}{2c^2}\frac{\vec{R_{12}}}{R_{12}^3} = \frac{1}{4\pi\varepsilon_0}\frac{I_1I_2}{c^2}\frac{\vec{R_{12}}}{R_{12}^3}$$

When the two wires are at right angles to each other, the contraction does not take place. Hence, the magnetic force is only due to the component of the current in wire 2 along that of wire 1. So we expect a force that looks like

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{I}_1\cdot\vec{I}_2}{c^2}\frac{\vec{R}_{12}}{R_{12}^3} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{I}_1\times\left(\vec{I}_2\times\vec{R}_{21}\right)}{c^2R_{21}^3}$$

where I have fudged - the last equality is only valid when $\vec{R}_{21} = -\vec{R}_{12}$ is perpendicular to $\vec{I}_1$ and $\vec{I}_2$. This is where the Biot-Savart Law comes from, and it also shows that $\mu_0 = 1/\varepsilon_0 c^2$. This derivation is only correct for infinite wires. The moment we move to $Id\vec{l}$, the "proof" above becomes merely a motivation.

The important lesson to take from this derivation is that magnetic field is nothing more than Coulomb's law together with relativity. It is not a new force.