First Test (25 Marks) **Apr 11, 2006**

Dept. of Electrical Engineering, IIT Madras
EC301 - Electromagnetic Fields

▷ **Derive your results.**

▷ **Draw diagrams wherever it makes sense.**

▷ **If you cannot evaluate some integral and get closed form answers, that does not matter. But I do want to see clear formulae.**

1. **A certain kind of electron beam of circular cross-section contains a current density of**

$$\vec{j} = j_0 \cos\left(\frac{\pi r}{2a}\right)\hat{z}$$

   (a) **Determine $\vec{B}(r,\theta,z)$ assuming $\mu = \mu_0$ and hence determine the magnetic force on the electrons.** .... **[5]**

      i. **$\vec{j}$ is along $\hat{z}$ and depends on r. Hence $\vec{A}$ is also along $\hat{z}$. Thus, since $\vec{A}$ depends only on r, $\vec{B}$ is along θ. ...1 mark**

      ii. **Using Stokes Law,**

$$\begin{aligned} 2\pi r B_\theta &= 2\pi\mu_0 \int_0^r j_0 \cos\left(\frac{\pi r'}{2a}\right) r'dr' \qquad \textbf{1 mark} \\ &= 2\pi\mu_0 \frac{2a}{\pi} j_0 \left[r\sin\left(\frac{\pi r}{2a}\right) + \frac{2a}{\pi}\left(1-\cos\left(\frac{\pi r}{2a}\right)\right)\right] \end{aligned}$$

      Thus,

$$\begin{aligned} B_\theta &= \frac{2\mu_0 j_0 a}{\pi r}\left[r\sin\left(\frac{\pi r}{2a}\right) + \frac{2a}{\pi}\left(1-\cos\left(\frac{\pi r}{2a}\right)\right)\right] \qquad \textbf{1 mark} \\ &= j_0 f(r) \qquad \textbf{for convenience} \end{aligned}$$

      iii. **The force on the electrons is given by**

$$\begin{aligned} \vec{j}\times\vec{B} &= -\hat{r} j_z B_\theta \\ &= -\frac{2\mu_0 j_0^2 a}{\pi r}\hat{r}\left[r\sin\left(\frac{\pi r}{2a}\right) + \frac{2a}{\pi}\left(1-\cos\left(\frac{\pi r}{2a}\right)\right)\right]\cos\left(\frac{\pi r}{2a}\right) \\ &= \mu_0 j_0^2 g(r)\,(-\hat{r}) \qquad \textbf{for convenience} \end{aligned}$$

      ...2 marks (1 mark for direction of $-\hat{r}$).

   (b) **Determine the spatial density profile of electrons, $n(r,\theta,z)$, that would be consistent with this profile. What velocity profile do the electrons need to be consistent with the current density itself?** .... **[5]**

      i. **The Electric Field balances the $\vec{j}\times\vec{B}$ force. So**

$$-ne\vec{E} = \mu_0 j_0^2 g(r)\hat{r} \qquad \textbf{1 mark}$$

i.e.,

$$E_r = -\frac{\mu_0 j_0^2 g(r)}{n(r)e} \quad \textbf{1 mark}$$

The divergence of this field gives $\rho/\varepsilon_0 = -en/\varepsilon_0$. **So**

$$\begin{aligned} n &= \frac{\mu_0\varepsilon_0}{e}\frac{j_0^2}{er}\partial_r\left\{\frac{rg(r)}{n(r)}\right\} \quad \textbf{1 mark} \\ n &= \frac{j_0^2}{e^2c^2}\left[\frac{1}{rn(r)}\partial_r\left(rg(r)\right) - \frac{g(r)}{n(r)^2}\partial_r n(r)\right] \\ \frac{j_0^2}{e^2c^2}\frac{g(r)}{n}\partial_r n + n^2 &= \frac{j_0^2}{e^2c^2}\frac{1}{r}\partial_r\left(rg(r)\right) \quad \textbf{1 mark} \end{aligned}$$

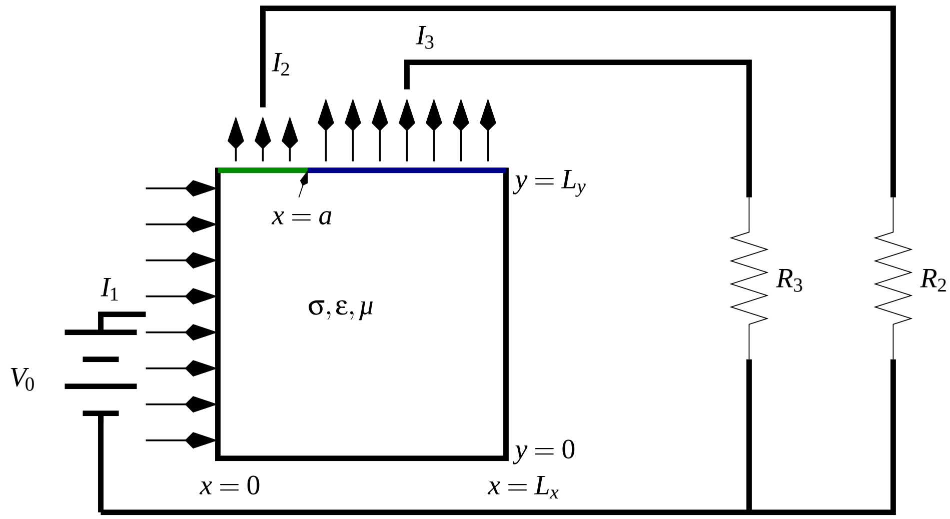
Note that $g(r)$ **is the dimension of** r, **while** j_0^2/e^2c^2 **has the dimension of** n^2. **So this equation is dimensionally correct. This is a nonlinear equation and has to be solved to get** n. **The velocity required is then given by**

$$j_z = -env_z$$

i.e.,

$$v_z = -\frac{j_z}{en} \quad \textbf{1 mark}$$

2. **Current flows into a square junction and flows out through two leads as shown in the figure below. The problem is to compute the current in resistor** R_2.



You may assume that the voltage at each lead is uniform over that lead. i.e., $\phi(0,y) = V_0$, $\phi(x,L_y) = V_2$ **for** $0 < x < a$ **and** $\phi(x,L_y) = V_3$ **for** $a < x < L_x$.

(a) **Solve Laplace's Equation in the junction and obtain the potential as a function of** V_1, V_2 **and** V_3. **............ [8]:**
The boundary conditions for potential are voltage specifications along the left and top walls and zero current specifications on the bottom and right walls. (1 mark)

We solve for the left wall first (top wall grounded):

$$\phi_1(x,y) = \frac{\cosh\left(k(L_x - x)\right)}{\cosh kL_x}\cos(ky) \quad \textbf{2 marks}$$

The potential has to go to zero on the top wall, which requires

$$kL_y = (n + 1/2)\pi \quad \textbf{1 mark}$$

i.e.,

$$k_n = \frac{(n+1/2)\pi}{L_y}$$

So

$$\phi_1 = \sum_{n=0}^{\infty} c_n \frac{\cosh\left(k_n(L_x - x)\right)}{\cosh\left(k_n L_x\right)}\cos(k_n y) \quad \textbf{1 mark}$$

On the left wall we have

$$V_0 = \sum_{n=0}^{\infty} c_n \cos(k_n y)$$

Solving we obtain (since we have a guarantee that the different $\cos$ **terms are orthogonal**

$$c_n = \frac{V_1 \int_0^{L_y}\cos(k_n y)\,dy}{\int_0^{L_y}\cos^2(k_n y)\,dy} \quad \textbf{1 mark}$$

Similarly for the top wall, we obtain

$$\phi_2 = \sum_{n=0}^{\infty} d_n \frac{\cosh\left(l_n(L_y - y)\right)}{\cosh\left(l_n L_y\right)}\cos(l_n x) \quad \textbf{1 mark}$$

where $l_n = (n+1/2)\pi/L_x$ **and have**

$$d_n = \frac{V_2 \int_0^{a}\cos(l_n x)\,dx + V_3 \int_a^{L_x}\cos(l_n x)\,dx}{\int_0^{L_x}\cos^2(l_n x)\,dx} \quad \textbf{1 mark}$$

(b) Determine the currents I_1, I_2 and I_3 that result from your solution. .... [4]:
The currents are got from the Electric Field, i.e, the current I_2 is given by

$$\begin{aligned} I_2 &= \int_0^a j_y dy \\ &= -\sigma \int_0^a \partial_y(\phi_1 + \phi_2) \end{aligned}$$

Similarly I_3 is given by

$$\begin{aligned} I_3 &= \int_a^{L_x} j_y dy \\ &= -\sigma \int_a^{L_x} \partial_y(\phi_1 + \phi_2) \end{aligned}$$

Note that all three currents are linear combinations of V_1, V_2 and V_3. Further, from the divergence theorem for current,

$$I_1 + I_2 + I_3 = 0$$

From the uniqueness theorem we have that the solution for $V_1 = V_2 = V_3 = V_0$ is $\phi_0(x,y) = V_0$ everywhere. This corresponds to a solution with no current, as there are no gradients. The given problem can be split into two parts, ϕ_0 and ϕ_2, where ϕ_0 is the function above, while ϕ_2 satisfies zero potential on the left wall and $V_2 - V_1$ and $V_3 - V_1$ on the top wall. The solution becomes $\phi(x,y) = \phi_0(x,y) + \phi_2(x,y)$. Only the d_n coefficients (defined in part 2a above) contribute to the current and we can write

$$\begin{aligned} I_2 &= -\sigma \int_0^a \partial_y (\phi_0 + \phi_2) \\ &= -\sigma \sum_{n=0}^{\infty} l_n d_n \frac{\sinh(l_n L_y)}{\cosh(l_n L_y)} \int_0^a \cos(l_n x)\, dx \\ &= -\sigma \sum_{n=0}^{\infty} l_n d_n \tanh(l_n L_y) \int_0^a \cos(l_n x)\, dx \end{aligned}$$

where the d_n are given in the previous part as

$$d_n = \frac{(V_2 - V_1)\int_0^a \cos(l_n x)\, dx + (V_3 - V_1)\int_a^{L_x} \cos(l_n x)\, dx}{\int_0^{L_x} \cos^2(l_n x)\, dx}$$

Thus,

$$I_2 = \alpha(V_2 - V_1) + \beta(V_3 - V_1)$$

where α and β can be read off the expression for I_2. Similarly for I_3. The general solution therefore becomes

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} V_2 - V_1 \\ V_3 - V_1 \end{pmatrix} = \begin{pmatrix} I_2 \\ I_3 \end{pmatrix}$$

The G_{ij} coefficients can be read off from the current expression, and have the form

$$\begin{aligned} G_{11} &= \sum_{n=0}^{\infty} \Lambda_n \int_0^a \cos(l_n x)\, dx \int_0^a \cos(l_n x)\, dx \\ G_{22} &= \sum_{n=0}^{\infty} \Lambda_n \int_a^{L_x} \cos(l_n x)\, dx \int_a^{L_x} \cos(l_n x)\, dx \\ G_{12} = G_{21} &= \sum_{n=0}^{\infty} \Lambda_n \int_0^a \cos(l_n x)\, dx \int_a^{L_x} \cos(l_n x)\, dx \end{aligned}$$

where

$$\Lambda = -\sigma l_n \tanh(l_n L_y) \Big/ \int_0^{L_x} \cos(l_n x)\, dx$$

and $l_n = (n + 1/2)\pi / L_x$. G_{ij} is symmetric as required by reciprocity.

(c) Obtain I_2 by solving the circuit problem. .... **[3]:**

This is just simple circuit theory, now that we have the admittance matrix.