Pulse Propagation in $1-D$

11th March 2007

Maxwell's Equations gave us the Wave Equation:

$$\begin{aligned}\nabla\times\left(\nabla\times\vec{E}\right) &= -\nabla\times\left(\partial_t\vec{B}\right)\\ \nabla\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E} &= -\mu\partial_t\left(\nabla\times\vec{H}\right)\\ &= -\mu\partial_t\vec{j}+\mu\varepsilon\partial_t^2\vec{E}\end{aligned}$$

Let us consider a medium in which μ and ε are spatially uniform. Then $\nabla\cdot\vec{E}=\nabla\cdot\vec{D}/\varepsilon=0$. If we further ignore currents (the medium is assumed insulating), we obtain the familiar wave equation

$$\nabla^2\vec{E}-\frac{1}{c^2}\partial_t^2\vec{E}=0$$

As I discussed in class, this is a swindle, since material response is *always* frequency dependent, i.e., time dependent. Thus, the correct version of this equation is

$$\nabla^2\vec{E}(\vec{r},\omega)+\frac{\omega^2}{c^2}\vec{E}(\vec{r},\omega)=0$$

or, in time domain, taking the inverse fourier transform,

$$\nabla^2\vec{E}(\vec{r},t)-\mu_0\partial_t^2\int_{-\infty}^{t}\vec{E}(\vec{r},t')\varepsilon(t-t')dt'=0$$

Here, $\varepsilon(t-t')$ is the impulse response of the atoms in the medium. Clearly we have a description of a linear, time-invarient channel here. Since the real world is neither linear nor time-invarient, even this equation is an approximation, but it is a very good one.

Another assumption has been made here, that the induced currents at a point $\vec{r}$ are due entirely to the applied field at that point. Otherwise, we would have a convolution in space as well. This is also a very good approximation in practice, though it gets questionable when we start analysing the electromagnetic properties of polymers and proteins, etc.

We now specialize to the 1-D case and obtain

$$\partial_z^2E_x-\mu_0\partial_t^2\int_{-\infty}^{t}E_x(t')\varepsilon(t-t')dt'=0$$

In this equation, we have assumed that μ is just the vacuum permeability. To solve this problem, we go to ω-k space:

$$E_x=\int\int\tilde{E}_x(k,\omega)e^{j\omega t}e^{-jkz}d\omega dk$$

Transforming in z, we obtain

$$-k^2\tilde{E}_x(k,t)-\mu_0\partial_t^2\int_{-\infty}^{\infty}\tilde{E}_x(k,t')\varepsilon(t-t')dt'=0$$

Transforming in t now yields

$$-k^2\tilde{E}_x(k,\omega)+\omega^2\mu_0\varepsilon(\omega)\tilde{E}_x(k,\omega)=0$$

Clearly,

$$k^2=\omega^2\mu_0\varepsilon(\omega)$$

must be satisfied if $\tilde{E}_x$ is to be non-zero. The integral becomes

$$\begin{aligned}E_x(z,t) &= \int\int\tilde{E}_x(k,\omega)e^{j\omega t}e^{-jkz}\left(\delta(k-\omega\mu_0\varepsilon)+\delta(k+\omega\mu_0\varepsilon)\right)d\omega dk\\ &= \int_{-\infty}^{\infty}\left[Ae^{j(\omega t-k(\omega)z)}+Be^{j(\omega t+k(\omega)z)}\right]d\omega\end{aligned}$$

where A and B are the amplitudes of the field along the two curves $k=\pm\omega\mu_0\varepsilon(\omega)$.

Suppose now that an antenna at $z=0$ sends out only the positive going wave. Hence, $B=0$, and from the field near the antenna, we have

$$E_x(0,t) = f(t) = \int_{-\infty}^{\infty} A e^{j\omega t} d\omega$$

Clearly this is a Fourier transform, and we can solve for $A(\omega)$ as

$$A(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

The wave propagates through all of space, and is given by

$$E_x(z,t) = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t')e^{-j\omega t'}dt'\right] e^{j(\omega t - k(\omega)z)} d\omega$$

A receiver antenna is placed at $z=L$, and we wish to know what is received. We find,

$$E_x(L,t) \quad = \quad \int_{-\infty}^{\infty} A(\omega) e^{j(\omega t - k(\omega)L)} d\omega$$

We expand $k(\omega)$ about the central frequency ω_c of the original signal $f(t)$. Then, $k = k_0 + k_0'(\omega-\omega_c) + k_0''(\omega-\omega_c)^2/2$ in the vicinity of ω_c. The field becomes,

$$\begin{aligned} E_x(L,t) &= e^{j(\omega_c t - k_0 L)}\int_{-\infty}^{\infty} A(\omega) e^{-jk_0''(\omega-\omega_c)^2 L} e^{j\left((\omega-\omega_c)t - k_0'(\omega-\omega_c)L\right)} d\omega \\ &\quad + e^{-j(\omega_c t - k_0 L)}\int_{-\infty}^{\infty} A(-\omega) e^{-jk_0''(\omega+\omega_c)^2 L} e^{j\left((-\omega+\omega_c)t - k_0'(-\omega+\omega_c)L\right)} d\omega \end{aligned}$$

Actually, the integrals is assumed to have significant power in the vicinity of ω_c and $-\omega_c$, respectively. The second integral represents the negative frequency branch, and is nothing but the complex conjugate of the positive branch, in order to keep E_x pure real. Thus, we can drop one of the integrals and write

$$E_x(L,t) = 2\mathrm{Re}\left\{ e^{j(\omega_c t - k_0 L)}\int_{-\infty}^{\infty} A(\omega) e^{-jk_0''(\omega-\omega_c)^2 L} e^{j\left((\omega-\omega_c)t - k_0'(\omega-\omega_c)L\right)} d\omega \right\}$$

Now, from the fourier transform of a Gaussian, we know that

$$e^{-jk_0''(\omega-\omega_c)^2 L} \rightleftharpoons C e^{-j\omega_c t} e^{-jt^2/4k_0''L}$$

where C is the appropriate constant required to conserve energy. Hence, the Electric Field in time is given by

$$E_x(L,t) = 2\mathrm{Re}\left\{ e^{-jk_0 L}\int_{-\infty}^{\infty} f(t') e^{-j\left(t - k_0'L - t'\right)^2/4k_0''L} dt' \right\}$$

Thus the received Electric Field is nothing but a delayed convolution of the originating Field with a very peculiar filter. Since the function $f(t)$ itself is pure real, we can now eliminate the "Real Part" operator and write

$$E_x(L,t) = 2\int_{-\infty}^{\infty} f(t')\cos\left(k_0 L + \left(t - k_0'L - t'\right)^2/4k_0''L\right) dt'$$

Now the function $f(t)$ has not been specified. Let us specify it now as a DSBSC AM modulation of a carrier wave by a gaussian pulse

$$f(t) = e^{-t^2/2}\cos(\omega_c t)$$

Then,

$$\begin{aligned} E_x(L,t) &= 2\int_{-\infty}^{\infty} e^{-t'^2/2}\cos(\omega_c t')\cos\left(k_0 L + \left(t - k_0'L - t'\right)^2/4k_0''L\right) dt' \\ &= \int_{-\infty}^{\infty} e^{-t'^2/2}\left[\cos\left(\omega_c t' + k_0 L + \left(t - k_0'L - t'\right)^2/4k_0''L\right)\right. \\ &\qquad \left. + \cos\left(\omega_c t' - k_0 L - \left(t - k_0'L - t'\right)^2/4k_0''Ldt'\right)\right] dt' \end{aligned}$$

Suppose now that the channel is non-dispersive. Then $k_0''=0$. Also assume that there is no carrier wave, i.e., $\omega_c=0$. Then, the field becomes

$$\begin{aligned} E_x(L,t) &= 2\int_{-\infty}^{\infty} e^{-t'^2/2}\cos\left(k_0 L + (t - k_0'L - t')^2/4k_0''L\right) dt' \\ &= 2\int_{-\infty}^{\infty} e^{-\left(u + t - k_0'L\right)^2/2}\cos\left(k_0 L + \alpha u^2\right) du \end{aligned}$$

where $\alpha = 1/4k_0''L$. The non-dispersive limit corresponds to $k_0'' \to 0$, i.e., $\alpha \to \infty$. Let us look at what happens to the pulse for different values of α at time $t - k_0'L$, and $k_0 L = 0$ (this is just a phase factor and is meaningless).

2 ⟨* 2⟩≡

```
alpha=[.1 .3 1 3];
```

We define the time vector (`t1` is used for the plot of the filters):

3a ⟨* 2⟩+≡
```
t=linspace(-5,5,101)';
t1=linspace(-5,5,501)';
```

Now for some function definitions. We define the integrand as a function and then call *intg* to solve for the Electric Field in time.

3b ⟨* 2⟩+≡
```
deff("y=integrnd(u,curt)","y=2*f(u+curt).*cos(curalpha*u.^2)");
deff("I=E(x)",["curt=x";"I=intg(-5,5,integrnd)"]);
deff("y=f(t)","y=exp(-(t+1.5).^2)+exp(-(t-1.5).^2)");
```

Create the array to hold *Efld* ahead of time, and define $f = e^{-t^2/2}$, which is the original pulse.

3c ⟨* 2⟩+≡
```
n=length(alpha);
Efld=zeros(length(t),n);
```

Obtain the output pulse for each value of $\alpha = 1\big/4k_0''L$. Note that there is a coefficient that I don't bother calculating. Instead I just use Parseval's theorem and equate the energy in the input and output pulses, since the dispersive filter is a unity gain filter. The results are plotted in subplots, where the left plots are the output (and input) pulses, while the right plots show the corresponding filter.

3d ⟨* 2⟩+≡
```
for i=1:n
  curalpha=alpha(i);
  Efld(:,i)=feval(t,E);
  Efld(:,i)=Efld(:,i)*norm(f(t))/norm(Efld(:,i));
  subplot(n,2,2*i-1);
  plot2d(t,[Efld(:,i) f(t)],[1 2]);
  xgrid(5);
  xtitle("alpha="+string(alpha(i)));
  subplot(n,2,2*i);
  plot2d(t1,cos(curalpha*t1.^2));
  xgrid(5);
  xtitle("cos("+string(curalpha)+"*t^2)");
end
```

Figure 1 shows the result of running the above code for a single transmitted pulse. As can be seen, as α decreases from a large value, the pulse spreads. This corresponds to increasing the link distance L. The computations are not very accurate since the filter is not fully captured in $(-5,5)$ which is the integral computed above.

Figure 2 shows what happens when $f(t) = u_0(t)$.

The lessons to take from these pictures are:

1. Despite the fact that the filter is *not* a conventional low pass filter in the sense of having lower absolute gain at higher frequencies, in practice it behaves like one.

2. The filter both low pass filters and adds oscillations to the pulse.

3. Since $\alpha \propto 1\big/k_0''L$, α starts at ∞ and continuously reduces till it approaches zero as the link length increases.

4. α has the dimensions of ω^2. The critical value of α where it starts having observable effects is when ω becomes comparable with ω_M, the modulation frequency.

What happens when we have two pulses? That can be seen in Figure 3. It is clear that there is a lot of crosstalk between the pulses when dispersion is present. This is a major reason why we don't want dispersion in communication channels.

Note: One final thing to remember: There is *no* loss of signal during dispersion. The signal phase has got mixed up. But it is all there. Which means that dispersion can be corrected. This is what is done in communications receivers where the mangled signal is put through an *adaptive filter* that undoes what the channel did and recovers the original pulse from the mess that was received.

Figure 1: Received pulse for different α for a transmitted gaussian pulse $e^{-t^2/2}$

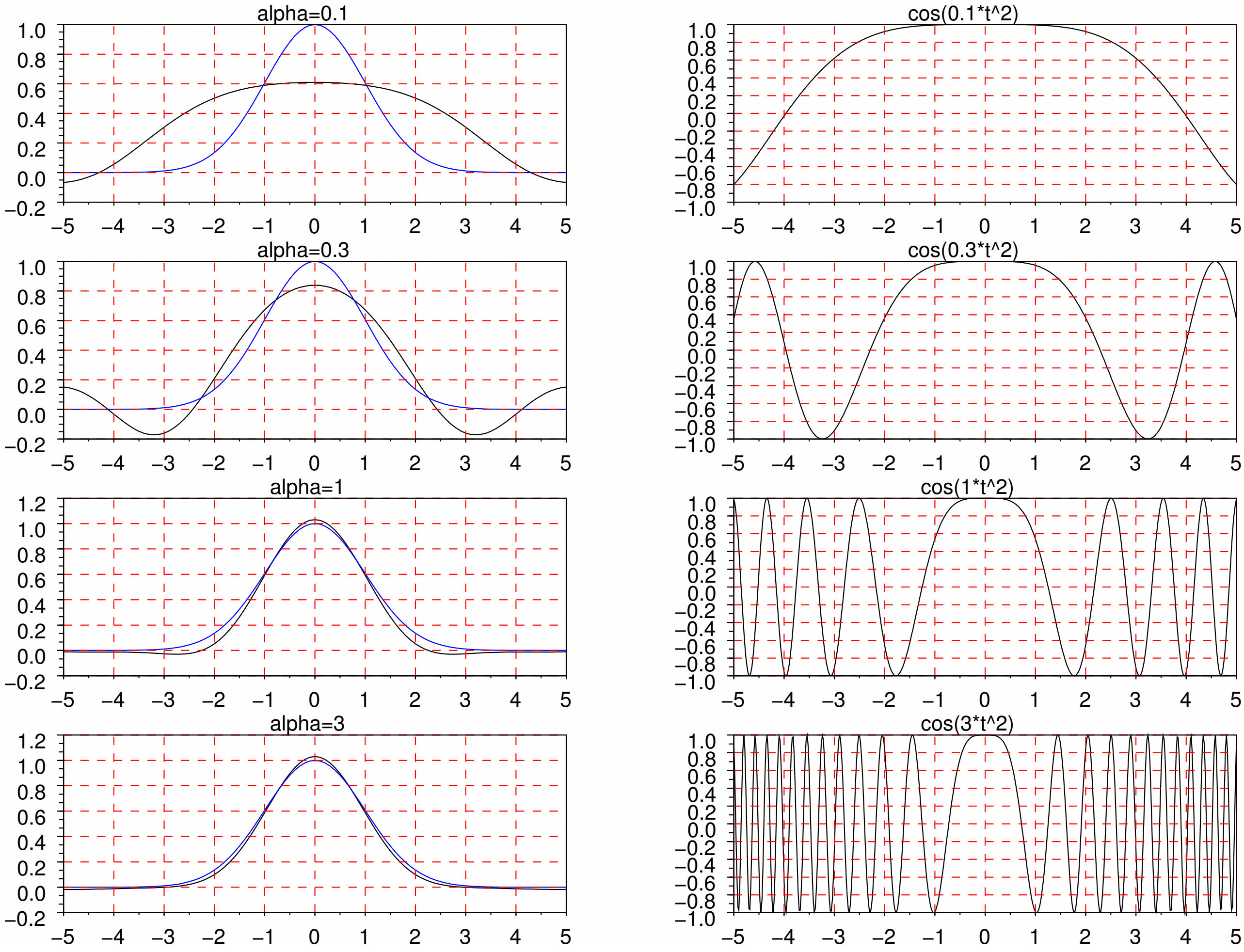
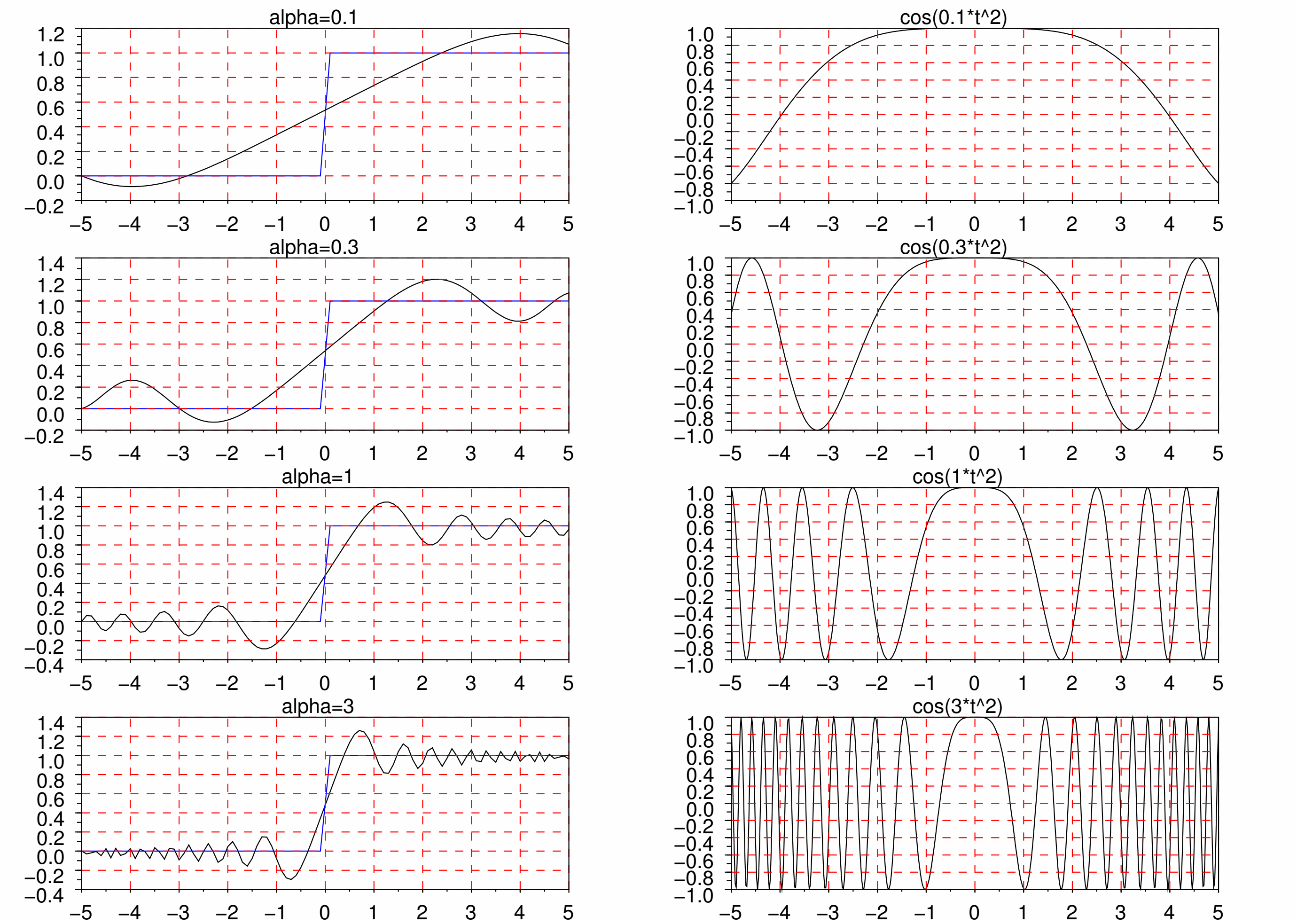


Figure 2: Effect of dispersion on a rising edge



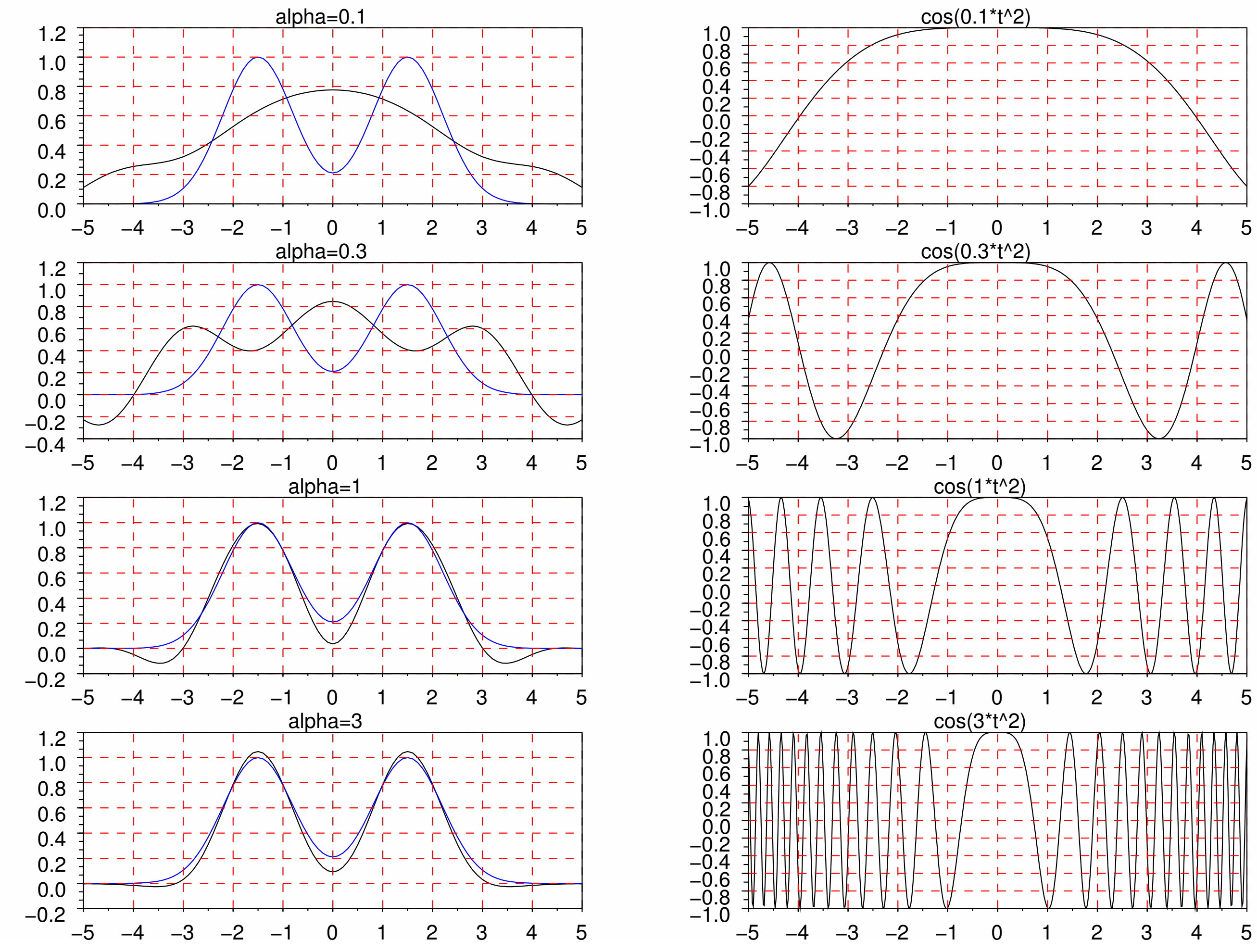


Figure 3: Received signal when two gaussian pulses are transmitted: $f(t) = e^{(t-1.5)^2} + e^{(t+1.5)^2}$