Problems in Magnetostatics

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Some of the later problems are quite challenging. This is characteristic of problems in magnetism. There are trivial problems and there are tough problems. Very few problems lie in between.

1. Which of these Magnetic fields can exist? Determine the current density that created the valid fields. What Vector Potential corresponds to these fields?

   (a) $\vec{B}(\vec{r}) = e^{-y^2}\hat{x}$:
   Divergence of the vector gives
   $$\nabla \cdot \vec{B} = \partial_x e^{-y^2} = 0$$
   The field does not blow up anywhere. Hence this is a valid magnetic field. (It does not go to zero for large x but small $|y|$ though).
   The curl gives the current:
   $$\frac{1}{\mu_0}\nabla \times \vec{B} = -\frac{\hat{z}}{\mu_0}\partial_y e^{-y^2} = \frac{2ye^{-y^2}}{\mu_0}\hat{z}$$
   The vector potential can be found directly from $\vec{B} = \nabla \times \vec{A}$. Let $\vec{A} = A_y\hat{y}$. Then
   $$B_x = -\partial_z A_y$$
   Hence, $A_y = -ze^{-y^2}$, i.e.,
   $$\vec{A} = -ze^{-y^2}\hat{y}$$

   (b) $\vec{B}(\vec{r}) = e^{-x^2}\hat{x}$:
   Divergence of this field is not zero. Hence it cannot be a valid magnetic field.

   (c) $\vec{B}(\vec{r}) = \sin(kr)\,\hat{r}$ (r, θ, z coordinate system):
   Again, divergence of this field is
   $$\nabla \cdot \vec{B} = \partial_r \sin(kr) = k\cos(kr)$$
   is not zero. Hence not a valid field.

(d) $\vec{B}(\vec{r}) = r\hat{\theta}$ (r, θ, z coordinate system):
The divergence of this field is zero since B_θ does not depend on θ. The current density is given by

$$\vec{j} = \frac{\hat{z}}{\mu_0}\frac{1}{r}\partial_r\left(r^2\right) = \frac{2\hat{z}}{\mu_0}$$

From Stokes theorem $\oint \vec{B}\cdot\vec{dl} = 2\pi r^2 = \pi r^2 j_0\mu_0$ which confirms this answer. The vector potential is again easiest to get from $\vec{B}$:

$$\vec{A} = rz\hat{r}$$

2. The following current densities are given. Determine the Magnetic Fields.

(a) $\vec{j}(\vec{r}) = re^{-r}\hat{\theta}$:
From symmetry, we expect $\partial_z, \partial_\theta = 0$. So, $\vec{B} = B_z(r)\hat{z}$. From Ampere's Law, we have

$$\mu_0 re^{-r} = -\partial_r B_z$$

Hence, $B_z(r) = B_z(0) - \mu_0\int_0^r r'e^{-r'}dr'$. Since the total current is bounded and goes quickly to zero along $\hat{r}$, $\lim_{r\to\infty} B_z = 0$. Hence,

$$B_z(r) = \mu_0\int_r^\infty r'e^{-r'}dr'$$

The vector potential is similarly got from $\vec{B} = \nabla\times\vec{A}$. Clearly, $\vec{A} = A_\theta(r)\hat{\theta}$.

$$B_z(r) = \mu_0\int_r^\infty r'e^{-r'}dr' = \frac{1}{r}\partial_r\left(rA_\theta\right)$$

This can again be integrated to obtain $\vec{A}$. Note that the integrals are obtainable via integration by parts, but that is not the point of this course.

(b) $\vec{j}(\vec{r})$ is a square of current of 1 Ampere with its sides along the z axis, the y axis, $y = 1$ and $z = 1$. The current is flowing along $+\hat{z}$ up the $z-$axis. (It is sufficient to give the answers in terms of integrals, if you cannot simplify them).:
For any line current I extending from $\vec{r}_0$ to $\vec{r}_0 + L\hat{z}$, the Vector Potential is given by

$$\vec{A} = \frac{\mu_0}{4\pi}I\hat{z}\int_0^L \frac{d\zeta}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0-\zeta)^2}}$$

Similarly for a line current along $\hat{y}$,

$$\vec{A} = \frac{\mu_0}{4\pi}I\hat{y}\int_0^L \frac{d\zeta}{\sqrt{(x-x_0)^2+(y-y_0-\zeta)^2+(z-z_0)^2}}$$

Hence the total Vector Potential is given by

$$\begin{aligned}\vec{A} &= \frac{\mu_0 I}{4\pi}\hat{z}\int_0^1\left[\frac{1}{\sqrt{x^2+y^2+(z-\zeta)^2}} - \frac{1}{\sqrt{x^2+(y-1)^2+(z-\zeta)^2}}\right]d\zeta \\ &\quad + \frac{\mu_0 I}{4\pi}\hat{y}\int_0^1\left[\frac{1}{\sqrt{x^2+(y-\zeta)^2+(z-1)^2}} - \frac{1}{\sqrt{x^2+(y-\zeta)^2+z^2}}\right]d\zeta\end{aligned}$$

which can easily be expressed in closed form. The curl of this can now be taken.

The far field due to this (or any) loop, can be obtained by Taylor expansion (see *Far field of an arbitrary current loop* in the site)

$$\begin{aligned}\vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi}\int\frac{\vec{j}(\vec{r}')}{\sqrt{r^2+r'^2-2\vec{r}\cdot\vec{r}'}}dV' \\ &\simeq \frac{\mu_0}{4\pi}\frac{1}{r}\int\vec{j}(\vec{r}')\left(1+\frac{\vec{r}\cdot\vec{r}'}{r^2}\right)dV' \\ &= \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}\end{aligned}$$

The derivation is lengthy and is given separately. The important thing to note is that this formula now applies to any loop, even one not in the plane. The magnetic dipole moment is given by

$$\vec{m} = \oint \vec{r}' \times \vec{j}(\vec{r}')dV'$$

3. A current loop is elliptical in shape and lies in the $y-z$ plane with its major axis along $\hat{y}$ and its centre at the origin. What components of $\vec{A}$ and $\vec{B}$ are present along the x-axis? What components are present along the y-axis? **Hint:** If a function is symmetric about a point, its derivitive at that point goes to zero.:

   For each element of current in $z<0$, there is a symmetrical element with $z>0$. Combining them yields a net $\vec{j}$ along $\hat{z}$. Hence $\vec{A}$ is along $\hat{z}$ everywhere on the $x-y$ plane, including along the x- and y-axes. Thus, $\vec{B}$ has only x and y components.

$$\vec{B} = \hat{x}\partial_y A_z - \hat{y}\partial_x A_z$$

   By symmetry we expect $\vec{B}$ to be along $\hat{x}$ on the y-axis and along $\hat{x}$ on the x-axis (this is obviously true for the special case of a circular loop).

4. Consider the solution for $\vec{A}$ and $\vec{B}$ for a finite solenoid. Obtain the radial component of the $\vec{B}$ field near the axis. **Hint:** Use the divergence theorem and collect terms order by order in smallness and find the lowest order radial term.:

   From the derivation of the field for the finite solenoid, we have an expression for $\vec{B} = B_z(z)\hat{z}$ on axis. Let

$$\begin{aligned}B_r &= \sum_{k=0}^{\infty} B_{rk}(z)r^k \\ B_z &= \sum_{k=0}^{\infty} B_{zk}(z)r^k\end{aligned}$$

   We have expressions for B_{z0} and $B_{r0}=0$. Now apply the divergence theorem to a cylinder of height dz and radius r

$$\sum_{k=0}^{\infty}\left(B_{zk}(z+dz)-B_{zk}(z)\right)2\pi\frac{r^{k+2}}{k+2}+\sum_{k=0}^{\infty}B_{rk}(z)r^k 2\pi r dz = 0$$

$$
\begin{aligned}
\sum_{k=0}^{\infty} \frac{2\pi}{k+2} \left(\partial_z B_{zk}(z)\right) r^{k+2} + \sum_{k=0}^{\infty} 2\pi B_{rk}(z) r^{k+1} &= 0 \\
2\pi \sum_{k=0}^{\infty} r^{k+1} \left(\frac{1}{k+1} \partial_z B_{z,k-1} + B_{rk} \right) &= 0
\end{aligned}
$$

Working order by order in r, this yields

$$
\begin{aligned}
B_{r0} &= 0 \\
B_{r1} &= -\frac{1}{2} \partial_z B_{z0}
\end{aligned}
$$

etc. Using (from the solenoid problem)

$$
B_{z0} = \mu_0 n I_0 \left[\frac{L/2 - z}{\sqrt{(L/2-z)^2 + a^2}} - \frac{-L/2 - z}{\sqrt{(-L/2-z)^2 + a^2}} \right], \qquad r \ll a
$$

B_{r1} can now be calculated for any z.

5. A solid wire carries a D.C. current I. How is this current distributed? What happens to the $q\vec{v} \times \vec{B}$ force acting on conduction electrons?:

   Due to symmetry, $\vec{j}$ depends on r but not on θ or z. Applying divergence theorem to a cylinder of radius r and arbitrary height, we obtain that

$$
2\pi r\, j_r = -\frac{dQ_{\text{encl}}}{dt} = 0
$$

   Thus, $\vec{j} = j_\theta(r)\hat{\theta} + j_z(r)\hat{z}$, which automatically implies that $\vec{A}$ also lies in the θ-z plane (why? remember the circular loop derivation).

   Now inside a conductor we must have

$$
\vec{F} = q_e n \left(\vec{E} + \vec{v}_e \times \vec{B} \right) = q_e n \vec{E} + \vec{j} \times \vec{B}
$$

   or

$$
\vec{j} \propto \left(q_e n \vec{E} + \vec{j} \times \vec{B} \right) \tag{1}
$$

   Now, $\vec{A}$ lies in the θ-z plane and depends only on r. Hence $\vec{B} = \nabla \times \vec{A}$ also lies in the θ-z plane and depends only on r. So $\vec{j} \times \vec{B}$ is the cross product of two vectors, both of which are in the $\hat{\theta}$-$\hat{z}$ plane. The direction of this term is $\hat{r}$. Thus, Eq. 1 can be written as

$$
\begin{aligned}
j_r &= 0 \\
j_\theta &= q_e n \left(\vec{v}_e \times \vec{B} \right)_\theta = 0 \\
j_z &= \sigma E_z \hat{z}
\end{aligned}
$$

   This, in turn implies $\vec{A} = A_z(r)\hat{z}$ and $\vec{B} = B_\theta(r)\hat{\theta}$. Since the magnetic field is static, Faraday's law (which we do not officially know about yet) does not apply, and the Electric field remains curl free. What this means is that

$$
\int_0^L E_z dz = \frac{1}{\sigma} \int_0^L j_z dz = \text{Applied Voltage}
$$

is independent of r. Note that this last step would fail if $\vec{E} \neq -\nabla\phi$. Thus, $\vec{j} = j_z\hat{z}$ is uniform over the cross-section.

How is this possible? After all we have an $Id\vec{l} \times \vec{B}$ force that would tend to push the current towards the center of the wire (this is called a pinch effect, and is real). The answer is that static Electric field builds up so that at each radius r

$$E_r = v_z B_\theta$$

Then, the total force on conduction electrons becomes

$$\vec{F} = q_e \left(E_z\hat{z} + E_r\hat{r} + \vec{v}_e \times \vec{B} \right) = q_e E_z \hat{z}$$

It is as if the magnetic field never was.

Note: Earlier we used $\vec{j} = \sigma\vec{E}$ and here we are violating that. The actual equation is $\vec{j} = \sigma\left(\vec{E} + \vec{v} \times \vec{B}\right)$. Since $\vec{B}$ is along θ and $\vec{v}$ is along z only the r component sees this extra term. This equation is known as generalized Ohm's law.

6. A pipe of irregular cross-section carries a D.C. current I. Determine how the current is distributed.:

   This is a more complicated problem. The cross-section is not given to be axially symmetric in θ. But it *is* given to be a cylinder, i.e., $\partial_z = 0$.

   Inside the metal, the steady state momentum equation for electrons yields

$$0 = q_e \left(\vec{E} + \vec{v}_e \times \vec{B} \right) - m_e \nu_e \vec{v}_e$$

   Thus, the current $\vec{j}$ must satisfy

$$\vec{j} = n_e q_e \vec{v}_e = \frac{n_e q_e^2}{m_e \nu_e} \left(\vec{E} + \vec{v}_e \times \vec{B} \right) = \alpha \left(n_e q_e \vec{E} + \vec{j} \times \vec{B} \right)$$

   Since $\vec{j}$ is orthogonal to $\vec{j} \times \vec{B}$, the $\vec{j} \times \vec{B}$ term cannot cause $\vec{j}$. This is actually the immediate proof that no steadystate current can exist in the absence of an Electric Field. Let $\vec{E} = E_\parallel \hat{j} + \vec{E}_\perp$. Then,

$$\begin{aligned} 0 &= q_e n_e \vec{E}_\perp + \vec{j} \times \vec{B} \\ \vec{j} &= \sigma E_\parallel \hat{j} \\ \nabla \times \vec{B} &= \mu_0 \vec{j} \end{aligned}$$

   Now, it is quite possible that part of the applied Electric Field goes towards building up $\vec{E}_\perp$. However, even when that is so, it must be the case that $E_\parallel$ is the uncancelled portion of the applied (external) Electric Field.

   - But this means that $E_\parallel$ is along $\hat{z}$ since the applied field is along $\hat{z}$.
   - Then $\vec{j}$ must also be along $\hat{z}$ since $\vec{j} = \sigma E_\parallel \hat{j}$.
   - It immediately follows that $\vec{A} = A_z(x,y)\hat{z}$.
   - Taking the curl, then, $\vec{B} = B_x\hat{x} + B_y\hat{y}$.

To summarize:

$$
\begin{aligned}
\vec{j} &= j_z\hat{z} = \sigma E_z\hat{z} \\
\vec{A} &= A_z\hat{z} \\
\vec{B} &= B_x\hat{x} + B_y\hat{y}
\end{aligned}
$$

This solution requires that we be able to find a curl-free Electric Field that satisfies

$$\vec{E}_\perp + \frac{\vec{j}}{q_e n_e} \times \vec{B} = 0$$

A solution will exist if $\nabla \times \vec{E}_\perp = 0$:

$$
\begin{aligned}
\nabla \times \vec{E} &= \frac{1}{q_e n_e} \nabla \times \left(\vec{j} \times \vec{B}\right) \\
&= \frac{1}{q_e n_e} \left[\left(\nabla \cdot \vec{B}\right)\vec{j} - \left(\nabla \cdot \vec{j}\right)\vec{B}\right] \\
&= 0
\end{aligned}
$$

since both $\vec{j}$ and $\vec{B}$ are divergence free.

Thus we now have a complete answer to the problem, since a uniform $\vec{j}$ fully specifies all the currents present, which allows us to calculate $\vec{E}_\perp$ and $\vec{B}$ as well.

7. A conducting fluid is forced across a uniform magnetic field. Show that a voltage develops across the terminals (with area A, kept at $x = 0$ and $x = L$). Assume that the terminal at $x = 0$ is grounded. If a resistor R is connected across the terminals, determine how this potential relates to the velocity profile of the fluid, $\vec{u} = u_z(x)\hat{z}$ and the strength of the $\vec{B} = B_0\hat{y}$ field. You may assume $\vec{j} = \sigma\left(\vec{E} + \vec{u} \times \vec{B}\right)$.:

   The fluid has a conductivity σ and flows with a velocity $u_z(x)\hat{z}$ across the terminals. The terminals are asssumed to be at $x = 0$ (grounded) and $x = L$ (at some voltage V), and have an area A. A uniform magnetic field $B_0\hat{y}$ is present. The terminals are connected across a resistor R, and carry a current V/R. This current must be the same at all x positions by charge continuity. Hence,

$$j_x = \sigma\left(E_x - u_z B_0\right) = -\frac{I}{A}$$

   We can therefore calculate V:

$$
\begin{aligned}
V &= -\int_0^L E_x dx \\
&= -\int_0^L \left(\frac{I}{A\sigma} + u_z B_0\right) \\
&= \frac{IL}{A\sigma} - B_0 \int_0^L u_z \\
&= -IR
\end{aligned}
$$

   where the $-$ sign in the last term was due to the fact that the direction of current was into the positive terminal of the MHD battery. Hence,

$$V = -\frac{V}{R}\frac{L}{A\sigma} - B_0 \int_0^L u_z$$

i.e.,

$$V = \frac{-B_0 \int_0^L u_z dx}{1 + L/RA\sigma} = \frac{-B_0 \int_0^L u_z dx}{1 + R_i/R}$$

8. Show that the magnetic field cannot do any work on a particle, i.e., cannot change its energy.:

   The force equation is

   $$m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

   The work done by the applied force is given by

   $$\begin{aligned} \frac{dW}{dt} &= \vec{F} \cdot \vec{v} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \vec{v} \\ &= q\vec{E} \cdot \vec{v} \end{aligned}$$

   Hence, the magnetic field cannot directly do any work on a charged particle. It *can* however do work via an induced electric field.

9. Given

   $$\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$$

   determine $\vec{B}(\vec{r})$. Then obtain $\nabla \times \vec{B}$ and show that it is zero except at the origin.

10. An axially symmetric ($\partial_\theta = 0$) magnetic field is given by

    $$B_z = B_0 \left(1 + 2\frac{z^2}{1 + z^2}\right)$$

    A proton (charge e, mass m_p) has an initial position at the origin and an initial velocity of $\vec{v} = (3\hat{x} + \hat{z}) \times 10^4/\sqrt{10}$ metres per second.

    (a) Determine B_r and B_θ. Sketch them. This is what is called a "magnetic mirror" where the axial magnetic field is stronger at two ends of a magnetic field. Examples include the earth's magnetic field which is strongest near the ends, i.e., the poles and weaker in the middle which is high in the magnetosphere.

    (b) Let the intial velocity of the proton be 10^4 metres per second along $\hat{x}$. Solve for the trajectory of the proton in time. Remember that the equation to be solved is

    $$m_p \dot{\vec{v}} = e\vec{v} \times \vec{B}$$

    The proton moves in circles in a uniform magnetic field.

    (c) Consider the actual trajectory. Assume (as is quite valid) that the distance travelled along $\hat{z}$ is negligible during a single gyroperiod ($T = 2\pi/\Omega_p$, $\Omega_P = eB_z(z)/m_p$). At any given position in z, determine the force on the proton due to $\vec{v} \times \vec{B}$. Write down the equation for the evolution of $v_z(t)$.

    (d) Invoke conservation of energy (see previous problem) to obtain the modified $v_r(t)$. Compute $\mu = mv_\perp^2/2B_z$. How does it evolve in time? **Note:** μ is what is called an adiabatic invariant, and it is an extraordinarily important quantity that is instrumental in helping us understand waves in magnetised gases (eg., solar wind, ionosphere, solar corona etc.)