Total Internal Reflection

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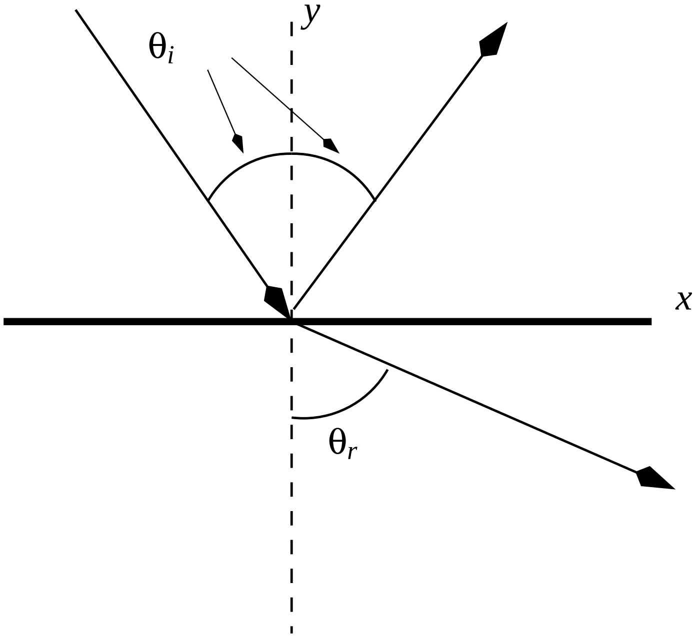
We have already worked out what happens when a wave is incident on a slab at an angle. One of the consequences is that $k_x = 2\pi/\lambda_x$ is the same for all three components.



This leads to Snell's Law, since

$$\sin\theta_i = \frac{k_x}{k_1} = \frac{ck_x}{\omega n_1}$$
$$\sin\theta_r = \frac{k_x}{k_2} = \frac{ck_x}{\omega n_2}$$

Hence,

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_2}{n_1}$$

where $n_{1,2} = c\sqrt{\mu\varepsilon}$ is the factor by which the speed of light has reduced in the medium below its vacuum value. Now consider a situation where n_2 (say air) is less than n_1 (say glass). Then,

$$\sin\theta_r = \frac{n_1}{n_2}\sin\theta_i$$

Clearly, for any given n_1 / n_2, there is a maximum θ_i beyond which a solution cannot be found. What happens if a wave comes in with a θ_i that is greater than this critical angle?

We consider a Transverse Electric polarization. Then the Electric Field is purely along $\hat{z}$ and is purely tangential to the dielectric surface. The magnetic field is partly normal, and partly tangential. The tangential component is $-E_{0z}\cos\theta_i / \eta$. The incident waves are given by

$$\begin{aligned}\vec{E}_i &= E_0 \hat{z} e^{j\left(\omega t - k_x x - k_y y\right)} \\ \vec{H}_i &= \frac{E_0}{\eta_1}\left(-\cos\theta_i \hat{x} - \sin\theta_i \hat{y}\right) e^{j\left(\omega t - k_x x - k_y y\right)}\end{aligned}$$

The reflected waves are given by

$$\begin{aligned}\vec{E}_r &= E_r \hat{z} e^{j\left(\omega t - k_x x + k_y y\right)} \\ \vec{H}_r &= \frac{E_r}{\eta_1}\left(\cos\theta_i \hat{x} - \sin\theta_i \hat{y}\right) e^{j\left(\omega t - k_x x + k_y y\right)}\end{aligned}$$

And the transmitted wave has the form

$$\begin{aligned}\vec{E}_t &= E_t \hat{z} e^{j\left(\omega t - k_x x - k'_y y\right)} \\ \vec{H}_t &= \frac{E_t}{\eta_2}\left(-\cos\theta_r \hat{x} - \sin\theta_r \hat{y}\right) e^{j\left(\omega t - k_x x - k'_y y\right)}\end{aligned}$$

Since ω and k_x are fixed by the incoming wave, the value of k'_y is determined by

$$k_y'^2 = \frac{\omega^2}{c^2} n_2^2 - k_x^2$$

Thus, $k_y'^2$ is negative, or k'_y is pure imaginary. Thus, the tramsitted wave can rather be written as

$$\vec{E}_t = E_t \hat{z} e^{j(\omega t - k_x x)} e^{k'_y y}$$

where $e^{-k'_y y}$ has been dropped since it would go to infinity at large negative y. Then, Faraday's Law yields

$$-j\omega\mu\vec{H}_t = E_t e^{j(\omega t - k_x x)} e^{k'_y y}\left(k'_y \hat{x} + j k_x \hat{y}\right)$$

So,

$$\eta\vec{H}_t = E_t e^{j(\omega t - k_x x)} e^{k'_y y}\left(\frac{k'_y \hat{x}}{jk} + \frac{k_x \hat{y}}{k}\right)$$

Let us now compute the Poynting flux in the second medium.

$$\begin{aligned}\text{Re}\left\{\vec{E}\times\vec{H}^*\cdot\hat{y}\right\} &= \text{Re}\left\{E_z H_x\right\} \\ &= = \left|E_t\right|^2 e^{2k'_y y}\text{Re}\left\{\frac{k'_y}{jk}\right\} \\ &= 0\end{aligned}$$

Thus, there is no power flowing into, or out of, the second medium, even though there are fields. *All the power is reflected*. There is, however a phase relationship between the incident and reflected wave, which can be worked out following the method described in the oblique incidence problem.