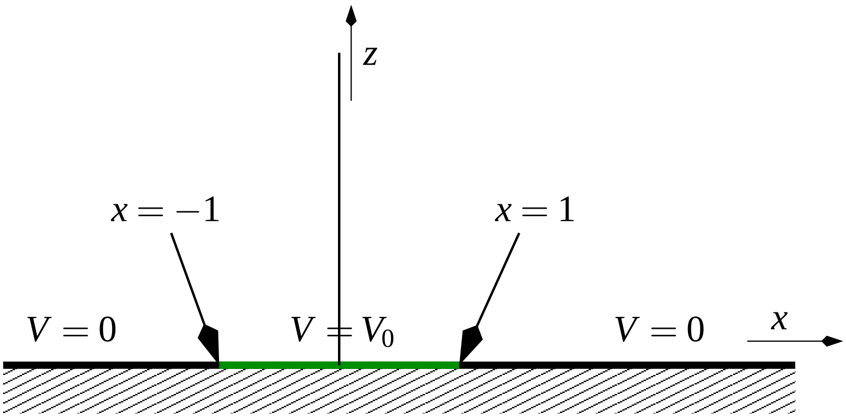
Use of Fourier and Laplace Transforms in Potential Problems

12th February 2007

Strip of Potential on a conducting plane

Consider a problem of an infinite plane that is grounded, except for a strip that is held at V_0 volts. We wish to find the potential for $z > 0$.



The x direction goes from $-\infty$ to ∞ and hence must use trigonometric solutions. The z direction has only positive values of z and exponential solutions are acceptible. Hence we can write down the general solution

$$\phi(x,z) = \int_{-\infty}^{\infty} c_k e^{jkx} e^{-|k|z} dk$$

Clearly we have a fourier transform in x. At $z = 0^+$, the potential is specified. Hence, we have

$$\phi(x,0^+) = \int_{-\infty}^{\infty} c_k e^{jkx} dk = V_0 u_0(1-x)$$

where a is the half-width of the strip of potential. This is trivially solved, via

$$\begin{aligned}
c_k &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(x,0^+)e^{-jkx}dx \\
&= \frac{V_0}{2\pi}\int_{-1}^{1} e^{-jkx}dx \\
&= -\frac{V_0}{2\pi}\frac{e^{jk}-e^{-jk}}{jk} \\
&= -\frac{V_0}{\pi}\frac{\sin k}{k}
\end{aligned}$$

Hence the potential for any x and any z is given by

$$\phi(x,z) = -\frac{V_0}{\pi}\int_{-\infty}^{\infty}\frac{\sin k}{k} e^{jkx} e^{-|k|z} dk$$

This can converted into a cosine integral by recognising the symmetry in k:

$$\phi(x,z) = -\frac{2V_0}{\pi}\int_{0}^{\infty}\frac{\sin k}{k}\cos(kx)\, e^{-kz} dk$$

Asymptotic Limits

The questions of interest are:

- How quickly does the potential decay if we move to large z for fixed x?
- How quickly does the potential decay if we move to large x for fixed z?

We can recognise that the integral is a Laplace Transform that takes k to z

$$\phi(x,z) = \mathcal{L}\left\{-\frac{2V_0}{\pi}\frac{\sin k}{k}\cos(kx), k \to z\right\}$$

For large z, it is the behaviour of the function at small k that matters. This can be Taylor expanded to give

$$\begin{aligned}
\phi(x,z) &\simeq \mathcal{L}\left\{-\frac{2V_0}{\pi}\left(1-\frac{k^2}{6}\right)\left(1-\frac{(kx)^2}{2}\right), k \to z\right\} \\
&\simeq \mathcal{L}\left\{-\frac{2V_0}{\pi}\left(1-k^2\left(\frac{1}{6}+\frac{x^2}{2}\right)\right), k \to z\right\} \\
&= -\frac{2V_0}{\pi}\left\{\frac{1}{z}-\frac{2}{z^3}\left(\frac{1}{6}+\frac{x^2}{2}\right)\right\}, \qquad 1, x \ll z
\end{aligned}$$

The potential decays as $1/z$. This should be compared to the potential of line charges that scale as $\ln z$. The change in scaling is due to the induced negative line charges in the ground plane.

To study the variation with x, we can recognise that $\cos kx$ is an oscillating function. Think of the integral as

$$\int_0^\infty f(k)\cos(kx)\,dk$$

Any $2\pi/x$ portion of the integral is basically equal to

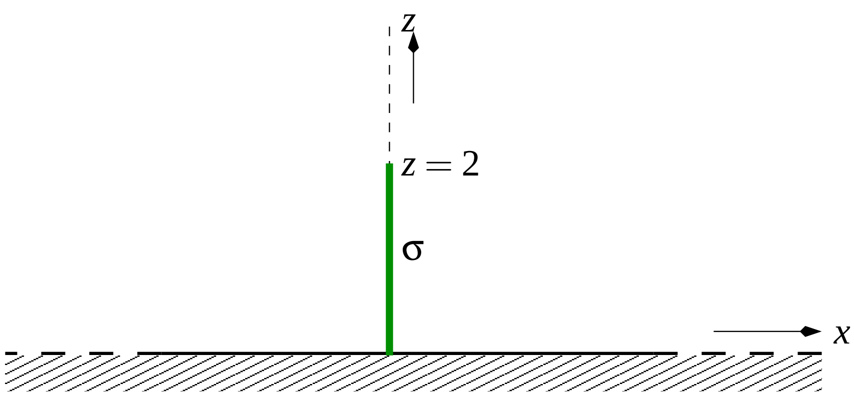
$$\begin{aligned}
\frac{\pi}{x}\left[-f\left(k+\frac{\pi}{x}\right)+\frac{1}{2}f(k)+\frac{1}{2}f\left(k+\frac{2\pi}{x}\right)\right] &= f'\left(k+\frac{3}{2}\frac{\pi}{x}\right)-f'\left(k+\frac{1}{2}\frac{\pi}{x}\right) \\
&= \frac{\pi}{x}f''\left(k+\frac{\pi}{x}\right)
\end{aligned}$$

Hence, the integral simplifies for large x to

$$-\frac{2V_0}{x}\int_0^\infty \partial_k^2\left(\frac{\sin k}{k}e^{-kz}\right)dk \propto \frac{1}{x}$$

Grounded Plane with Charge Sheet

Consider a conducting, grounded sheet in the x-y plane. A sheet of charge (σ Coulombs per square metre) is placed on the x-z plane for $0 < z < 2$. We want the potential for $x > 0$, $z > 0$.



As the problem is symmetric, there will be no E_x on the x-z in the region beyond $z = 2$. Thus along the z axis, the boundary condition is purely a specification of E_x.

What I mean here is that due to the boundary conditions and the applied charge being symmetric in x, the resulting potential is symmetric in x, i.e., $\phi(x,z)=\phi(-x,z)$ (y is assumed ignorable). Then,

$$E_x(0,z)=\lim_{x\to 0}\frac{\phi(x,z)-\phi(-x,z)}{2x}=0$$

provided the limit exists. For instance, the limit does not exist for $0<z<2$, since a charge sheet is present, and the gradient is discontinuous at $x=0$ - it is negative for $x>0$ and positive for $x<0$. Thus, $E_x=0$ beyond the charge sheet.

The condition along $z=0$ is given by $\phi(x,0^+)=0$. The condition along $x=0$ is $\partial_x\phi=-E_x=-\sigma/\varepsilon_0$. We try a solution of the form

$$\phi(x,y)=\phi(r,\theta)=F(r)G(\theta)$$

The equation becomes

$$\frac{r}{F}\partial_r\left(r\partial_r F\right)+\frac{1}{G}\partial_\theta^2 G=0$$

So,

$$\begin{aligned} r^2F''+rF'+k^2F &= 0 \\ \partial_\theta^2 G-k^2G &= 0 \end{aligned}$$

Then

$$G=\sinh k\theta$$

to account for the condition at $z=0$, and the equation for F yields

$$F=Cr^{jk}+Dr^{-jk}=Ce^{jk\ln r}+De^{jk\ln r}$$

The general solution in $u=\ln r$, θ becomes

$$\phi(u,\theta)=\int_{-\infty}^{\infty}c_k\frac{\sinh\left(k\theta\right)}{\cosh\left(k\pi/2\right)}e^{jku}$$

At $\theta=\pi/2$, the θ-derivitive yields the condition

$$\frac{1}{r}\partial_\theta\phi(r,\theta)=-E_\theta=\begin{cases}\sigma/\varepsilon_0 & 0<r<2\\ 0 & r>2\end{cases}$$

Thus the condition in u,θ becomes

$$\partial_\theta\phi(u,\theta)=\begin{cases}e^u\sigma/\varepsilon_0 & -\infty<u<\ln 2\\ 0 & u>\ln 2\end{cases}$$

Substituting into the integral this yields

$$\int_{-\infty}^{\infty}kc_ke^{jku}=\begin{cases}e^u\sigma/\varepsilon_0 & -\infty<u<\ln 2\\ 0 & u>\ln 2\end{cases}$$

This problem can now be solved in the standard way to obtain c_k and hence the potential everywhere.

An alternate way of solving the problem would have been to use

$$\phi(x,y)=\sin(ky)e^{-kx}$$

Then the general solution would have been

$$\phi(x,y)=\int_0^{\infty}c_k\sin\left(ky\right)e^{-kx}dk$$

The boundary condition at $y=0$ is automatically satisfied since $\sin\left(ky\right)$ goes to zero at $y=0$. The condition on charge now becomes

$$\int_0^{\infty}kc_k\sin\left(ky\right)dk=\begin{cases}\sigma/\varepsilon_0 & 0<y<2\\ 0 & y>2\end{cases}$$

This can be solved using the standard approaches.

A third approach is to use the method of images. Replace the ground plane by a sheet of *negative* charge from $z=-2$ to $z=0$. Now the potential can be calculated from the Gauss' Law result for a line charge. For a line charge at origin, we have

$$E_r=\frac{\lambda}{2\pi\varepsilon_0 r}$$

and

$$\phi=\frac{\lambda}{2\pi\varepsilon_0}\frac{r}{a}$$

where a is any reference distance at which we set potential to zero. Using this expression for the charge sheet we get

$$\phi(x,z)=\frac{\sigma}{2\pi\varepsilon_0}\int_0^2\left[\ln\left(x^2+\left(z-z'\right)^2\right)-\ln\left(x^2+\left(z+z'\right)^2\right)\right]dz'$$

Please Note: I messed up in the previous incarnation of this document. Thanks to you folks for pointing it out.

By the uniqueness theorem, all three approaches will give the same answer. Which answer is more convenient, or easier to compute depends on the problem. In the current case, obviously the method of images gives us a simpler answer.

The method of images is a special case of a general technique called Green's Functions. Given any problem, we solve the problem of placing a building block charge in that region, say a point charge, or a line charge or anything else, from which we can build the answer. We construct the answer by combining these building block answers. In the current example, our building block charges were line charges. We used the potential for that problem in an integral to obtain the potential for a sheet. Unfortunately, getting the Green's function can often be more complicated than solving the problem directly.

One final way to solve this problem is the way I mentioned as a general method earlier: Find a particular solution of Poisson's Equation and solve Laplace's Equation to match the modified boundary conditions. Here, that would mean we start with

$$\phi_P(x,z) = \frac{\sigma}{2\pi\varepsilon_0}\int_0^2 \ln\left(x^2 + \left(z-z'\right)^2\right) dz'$$

which is simply Coulomb's law for the given charge sheet. The boundary condition to be satisfied is $\phi(x,0^+) = \phi_P + \phi_H = 0$. So we now need to solve Laplace's Equation for ϕ_H with the boundary condition

$$\phi_H(x,0^+) = -\frac{\sigma}{2\pi\varepsilon_0}\int_0^2 \ln\left(x^2 + z'^2\right) dz'$$

Our solution now follows the standard approach:

$$\begin{aligned}
\phi_H(x,z) &= \int_{-\infty}^{\infty} c_k e^{jkx} e^{-kz} \\
\phi_H(x,0^+) &= \int_{-\infty}^{\infty} c_k e^{jkz} = -\frac{\sigma}{2\pi\varepsilon_0}\int_0^2 \ln\left(x^2 + z'^2\right) dz'
\end{aligned}$$

The c_k are found by solving the Fourier Transform problem and that gives us the full solution.

The lesson here is that there is no one way of solving these problems. *Any way is acceptible*. As long as it satisfies the equation and the boundary conditions, which expansion or which trick you used to get there does not matter.