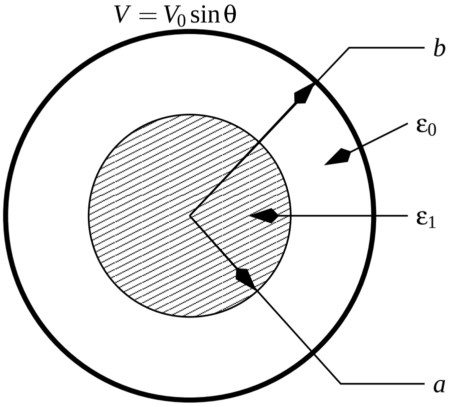
Laplace's Equation: Example

8th February 2007

The Problem



A cylinder is partially filled with a dielectric ε_1 with the rest of the volume being air. A voltage of $V_0 \sin\theta$ is applied at the wall ($r = b$). The problem is to find the potential within the cylinder.

Laplace's Equation in cylindrical coordinates is

$$\frac{1}{r}\partial_r (r\partial_r\phi) + \frac{1}{r^2}\partial_\theta^2\phi = 0 \tag{1}$$

You are expected to know this stuff. If you don't read Appendix 2 of the textbook, where the vector operators are "derived" in generalised coordinates.

We try the separation of variables approach and guess

$$\phi(r,\theta) = F(r)G(\theta)$$

Equation 1 now becomes

$$\frac{G}{r}\partial_r (r\partial_r F) + \frac{F}{r^2}\partial_\theta^2 G = 0$$

Multiplying by r^2/FG, we get

$$\frac{r}{F}\partial_r (r\partial_r F) + \frac{1}{G}\partial_\theta^2 G = 0$$

Since the first term depends only on r and the second only on θ, we can set them separately equal to a constant. To short circuit the next part, we can see that the system is periodic in θ, which means that G must be trigonometric in nature. The 2π periodicity implies

$$\begin{aligned} \partial_\theta^2 G + n^2 G &= 0 \\ r^2\partial_r^2 F + r\partial_r F - n^2 F &= 0 \end{aligned}$$

Solutions are of the form

$$G = A\cos n\theta + B\sin n\theta$$

and

$$F = Cr^\alpha + Dr^\beta$$

Here α and β must satisfy the characteristic equation

$$\alpha(\alpha-1) + \alpha - n^2 = 0$$

Clearly $\alpha = n$ and $\beta = -n$. So

$$F = Cr^n + Dr^{-n}$$

and for $n = 0$, we get

$$F = C + D\ln r$$

In the region $0 \le r < a$, the r^{-n} term is not acceptible and we get

$$\phi_1(r,\theta) = C_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n [C_n\cos\theta + D_n\sin\theta]$$

In the region $a < r \le b$, both radial terms are acceptible. We then get

$$\phi_2(r,\theta) = E_0 + \sum_{n=1}^{\infty} \left[E_n\left(\frac{r}{a}\right)^n + F_n\left(\frac{r}{a}\right)^{-n}\right][H_n\cos\theta + G_n\sin\theta]$$

One great simplification that we get is that the boundary potential has been given as $V_0\sin\theta$. Orthogonality means that only $G_1 = V_0$ is non-zero and all the other G and H coefficients are zero.

$$\phi_2(r,\theta) = \left(E_1\left(\frac{r}{a}\right) + F_1\left(\frac{a}{r}\right)\right)V_0\sin\theta \tag{2}$$

The boundary condition at $r = b$ requires

$$E_1\left(\frac{b}{a}\right) + F_1\left(\frac{a}{b}\right) = 1$$

At $r = a^+$, the potential becomes

$$\phi_2(a^+,\theta) = V_0(E_1 + F_1)\sin\theta$$

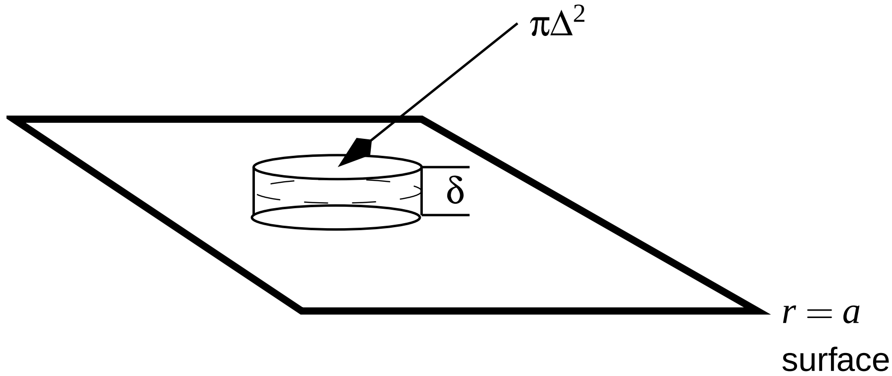
Let us see what happens at $r = a$. First, the potential must be continuous, since the field is bounded. This means that

$$V_0(E_1 + F_1)\sin\theta = C_0 + \sum_{n=1}^{\infty}[C_n\cos n\theta + D_n\cos n\theta]$$

But these are orthogonal functions and it is obvious that only D_1 is non-zero and it is given by $V_0\,(E_1+F_1)$. Thus,

$$\phi_1(r,\theta) = V_0\,(E_1+F_1)\left(\frac{r}{a}\right)\sin\theta \tag{3}$$

We are almost there. We need one more equation to pin down E_1 and F_1. Let us consider the displacement vector $\vec{D}$. Consider a cylinder radius Δ and height δ such that the $r=a$ surface cuts the cylinder.



Now we send Δ to zero, while keeping $\delta \ll \Delta$. Applying divergence theorem to this "pill box", we obtain

$$D_r(a^+,\theta) = D_r(a^-,\theta) \qquad \text{for all } \theta$$

i.e.,

$$-\varepsilon_0 \partial_r \phi_2(r,\theta)|_{r=a} = -\varepsilon_1 \partial_r \phi_1(r,\theta)|_{r=a}$$

Using Eq. 3 and Eq. 2 in this equation we obtain

$$-\varepsilon_0\left(\frac{E_1}{a}-\frac{F_1}{a}\right)V_0\sin\theta = -\varepsilon_1\left(\frac{E_1+F_1}{a}\right)V_0\sin\theta$$

i.e.,

$$\varepsilon_0\,(E_1-F_1) = \varepsilon_1\,(E_1+F_1)$$

Thus we have the following system of equations

$$\begin{pmatrix} b/a & a/b \\ 1 & -1 \end{pmatrix}\begin{pmatrix} E_1 \\ F_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon_1/\varepsilon_0 \end{pmatrix}$$

The solution is

$$\begin{pmatrix} E_1 \\ F_1 \end{pmatrix} = -\frac{1}{b/a-a/b}\begin{pmatrix} -1 & -a/b \\ -1 & b/a \end{pmatrix}\begin{pmatrix} 1 \\ \varepsilon_1/\varepsilon_0 \end{pmatrix} = \frac{1}{b/a-a/b}\begin{pmatrix} 1+a\varepsilon_1/b\varepsilon_0 \\ b\varepsilon_1/a\varepsilon_0 - 1 \end{pmatrix}$$

So we finally obtain the potential in the cylinder:

$$\phi(r,\theta) = \begin{cases} V_0\left(\frac{a}{b}+\frac{b}{a}\right)\frac{\varepsilon_1/\varepsilon_0}{b/a-a/b}\left(\frac{r}{a}\right)\sin\theta, & r<a \\ V_0\left(\frac{1+a\varepsilon_1/b\varepsilon_0}{b/a-a/b}\frac{r}{a}+\frac{b\varepsilon_1/a\varepsilon_0-1}{b/a-a/b}\frac{a}{r}\right)\sin\theta & r>a \end{cases} \tag{4}$$

Let us graph the field lines for $\varepsilon_1/\varepsilon_0 = 2.25$ (glass-air interface) with $a = 0.5b$.

3 ⟨∗ 3⟩≡

```
eps1=2.25;
a=0.5;
N=100; // set to even number to avoid atan singularties.
```

We work with a cartesian grid and set all points with r>1 to a potential of 1 Volt.

4a ⟨* 3⟩+≡
```
x=linspace(-1,1,N);
y=linspace(-1,1,N)';
phi=ones(N,1)*x.^2+y.^2*ones(1,N);
r=sqrt(phi);
theta=atan(y*ones(1,N),ones(N,1)*x);
phi(find(phi>1))=1;
indx1=find(phi<a^2);
indx2=find(phi<1 & phi>=a^2);
```

`indx1` contains the indices corresponding to points in region 1 while `indx2` contains indices corresponding to points in region 2. We now use Eq. 4 to compute $\phi(r,\theta)$.

4b ⟨* 3⟩+≡
```
phi(indx1)=(a+1/a)*eps1*r(indx1) ...
  .*sin(theta(indx1))/(1/a-a);
phi(indx2)=( (1+a*eps1)*r(indx2) ...
   +(eps1/a-1) ./r(indx2)) ...
   .*sin(theta(indx2))/(1/a-a);
xset("window",0);
clf;
xselect();
contour(x,y,phi,20);
```

The Electric field is along $\vec{y}$ for this case (Scilab plots $\hat{x}$ vertically and $\hat{y}$ horizontally) inside the inner region, and connects to the wall potential in the outer region.

The reason for the Electric field being cartesian in the inner region is that $r\sin\theta = y$. So $\phi(r,\theta) \propto y$ which means that the Electric field is uniform and along $\hat{y}$, in the inner region. But in the outer region, there is also a term that goes like $\sin\theta/r$, which is definitely not along x or y.

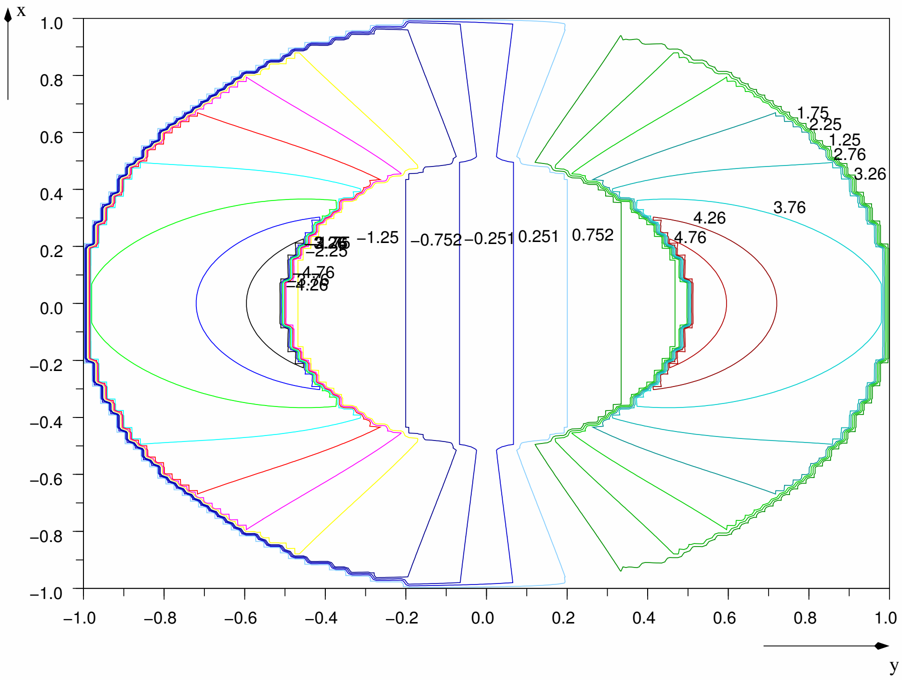


Figure 1: Potential contours of solution