Laplace's Equation: Example

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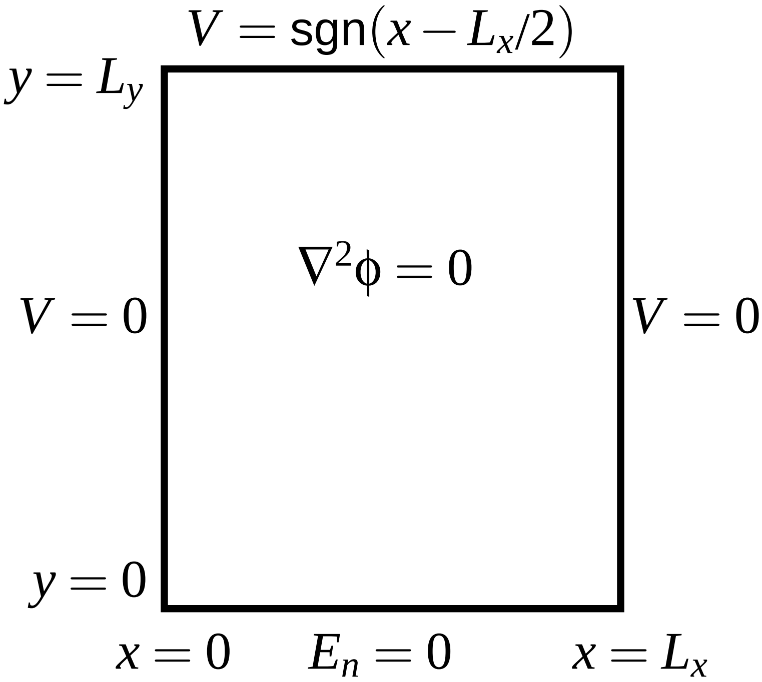
The Problem



A $2-D$ problem is solved here as an example of the use of the separation of variables approach. We wish to solve Laplace's Equation in a box of size L_x by L_y. The boundary conditions are:

- $V = 0$ on both side walls.
- $E_n = -\partial\phi/\partial y = 0$ on the bottom wall.
- $V = \mathrm{sgn}\left(x - L_x/2\right)$ on the top wall.

As in the class derivation, we look for solutions of the form $\phi(x,y) = F(x)G(y)$. Substituting into Laplace's Equation, we obtain equations for F and G:

$$\begin{aligned} F'' + k^2 F &= 0 \\ G'' - k^2 G &= 0 \end{aligned} \tag{1}$$

The sign of k^2

Depending on the sign of k^2, we obtain either trigonometric or exponential solutions. Since the RS of Laplace's Equation is zero, atleast one of the two equations in Eq. 1 must have a negative coefficient. Suppose k^2 is negative. Then

$$F(x) = Ae^{kx} + Be^{-kx}$$

We require $F(x)$ to be zero at $x = 0$ and at $x = L_x$, i.e.,

$$\begin{pmatrix} 1 & 1 \\ e^{kL_x} & e^{-kL_x} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

A and B can be non-zero only if the determinant of the matrix can go to zero. The determinant is given by

$$\det M = e^{-kL_x} - e^{kL_x} = -2\sinh(kL_x)$$

and can be zero only if $k = 0$ or $L_x = 0$, both of which are uninteresting special cases.

Suppose k^2 is positive. Then the equation becomes

$$F(x) = A\cos kx + B\sin kx$$

and the boundary conditions imply

$$\begin{pmatrix} 1 & 0 \\ \cos kL_x & \sin kL_x \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The determinant is now $\det M = \sin kL_x$ which can be zero whenever $k = n\pi/L_x$. That is why we choose k^2 positive in the rest of the derivation.

Obtaining the Solution

The solutions along x must go to zero at both $x = 0$ and at $x = L_x$. Hence

$$F(x) = \sin\left(\frac{n\pi}{L_x}x\right) \tag{2}$$

The solution along y must have zero derivitive at $y = 0$. Hence it can be chosen as

$$G(y) = \cosh\left(\frac{n\pi}{L_x}y\right) \tag{3}$$

Thus, the complete solution can be expressed as

$$\phi(x,y) = \sum_{n=1}^{\infty} c_n \cosh\left(\frac{n\pi}{L_x}y\right) \sin\left(\frac{n\pi}{L_x}x\right) \tag{4}$$

Evaluating at $y = L_y$ we obtain the final condition

$$\mathrm{sgn}\left(x - \frac{L_x}{2}\right) = \sum_{n=1}^{\infty} c_n \cosh\left(\frac{n\pi}{L_x}L_y\right) \sin\left(\frac{n\pi}{L_x}x\right)$$

Using orthogonality we obtain

$$
\begin{aligned}
c_m \cosh\left(\frac{m\pi}{L_x}L_y\right)\frac{L_x}{2} &= \int_0^{L_x} \operatorname{sgn}\left(x-\frac{L_x}{2}\right)\sin\left(\frac{m\pi}{L_x}x\right) \\
&= \begin{cases} 0 & m \text{ odd} \\ \left(1-(-1)^{m/2}\right)L_x\big/m\pi & m \text{ even} \end{cases}
\end{aligned}
$$

Thus, the final solution becomes

$$
\phi(x,y) = \sum_{n=1}^{\infty} \frac{L_x}{2n\pi}\left(1-(-1)^n\right)\frac{\cosh\left(2n\pi y\big/L_x\right)}{\cosh\left(2n\pi L_y\big/L_x\right)}\sin\left(\frac{2n\pi}{L_x}x\right) \tag{5}
$$

In summary, all problems reduce to finding basis functions, generating a sum, and then solving for the boundary potentials.