Uniqueness of field given its divergence and curl

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There is a simple and a complicated proof of this theorm, which is called Helmholtz theorem. It states that, given $\nabla \cdot \vec{V}$ and $\nabla \times \vec{V}$, the field $\vec{V}$ is uniquely determined. The simple proof is as follows.

Simple Version of Theorem

Let the $G(\vec{r})$ be curl of an unknown vector field and let $f(\vec{r})$ be the divergence, and suppose $\vec{V}_1$ and $\vec{V}_2$ be such that

$$\begin{array}{ll} \nabla \times \vec{V}_1 = G & \qquad \nabla \cdot \vec{V}_1 = f \\ \nabla \times \vec{V}_2 = G & \qquad \nabla \cdot \vec{V}_2 = f \end{array}$$

Consider $\vec{U} = \vec{V}_1 - \vec{V}_2$. Then

$$\begin{aligned} \nabla \times \vec{U} &= 0 \\ \nabla \cdot \vec{U} &= 0 \end{aligned}$$

Since the curl is zero everywhere, $\vec{U}$ is a conservative field and we can define

$$\phi(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{U} \cdot \vec{dl}$$

where ϕ does not depend on the path of the integration. Then $\vec{U} = -\nabla\phi$. The zero divergence of $\vec{U}$ then implies

$$\nabla^2\phi = 0$$

The boundary condition satisfied by $\vec{U}$ at ∞ is that it goes to zero, which implies $\phi \to 0$ as $r \longrightarrow \infty$. The unique solution to this problem is $\phi \equiv 0$, i.e., $\vec{U} \equiv \vec{0}$, i.e., $\vec{V}_1 = \vec{V}_2$.