Energy in the Magnetic Field

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Consider a current distribution $\vec{j}_0(\vec{r})$ that results in a magnetic field $\vec{B}_0(\vec{r})$. We wish to determine the energy in the field.

We raise the current density from zero to its full value uniformly:

$$\vec{j}(\vec{r},t) = \vec{j}_0(\vec{r})\alpha(t)$$

where $\alpha(t)$ is an arbitrary function that is zero at $t = 0$ and is unity at $t = T$. Now,

$$\begin{aligned}
\vec{B}(\vec{r},t) &= \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{r}') \times \vec{R}_{12}}{R_{12}^3} dV' \\
&= \alpha(t)\left[\frac{\mu_0}{4\pi}\int \frac{\vec{j}_0(\vec{r}') \times \vec{R}_{12}}{R_{12}^3} dV'\right] \\
&= \alpha(t)\vec{B}_0(\vec{r})
\end{aligned}$$

So, the Magnetic Field has its final shape at all times, but its amplitude is modulated by the function $\alpha(t)$. This changing magnetic field induces an Electric Field.

$$\nabla \times \vec{E} = -\partial_t \vec{B} = \frac{d\alpha}{dt}\vec{B}_0(\vec{r})$$

The work done *by* the Electromagnetic Fields is given by

$$\begin{aligned}
W &= \int_0^T \vec{F} \cdot \vec{v} dt \\
&= \int_0^T \left[\int \rho\left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \vec{v} dV\right] dt \\
&= \int_0^T \left[\int \rho \vec{E} \cdot \vec{v} dV\right] dt \\
&= \int_0^T \left[\int \vec{E} \cdot \vec{j} dV\right] dt
\end{aligned}$$

Using Ampere's Law this becomes

$$\begin{aligned}
W &= \int_0^T \left[\int \vec{E} \cdot \nabla \times \vec{H} dV\right] dt \\
&= \int_0^T \left[\int \varepsilon_{ijk} E_i \partial_j H_k dV\right] dt \\
&= \int_0^T \left[\int \left\{\varepsilon_{ijk}\partial_j\left(E_i H_k\right) - \varepsilon_{ijk} H_k \partial_j E_i\right\} dV\right] dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \left[\int \left\{ -\nabla \cdot \left(\vec{E} \times \vec{H} \right) + \vec{H} \cdot \nabla \times \vec{E} \right\} dV \right] dt \\
&= -\int_0^T \left[-\oint \vec{E} \times \vec{H} \cdot d\vec{S} + \int \vec{H} \cdot \partial_t \vec{B} dV \right] dt \\
&= -\int_0^T \left[\int \alpha(t) \vec{H}_0 \cdot \frac{d\alpha}{dt} \vec{B}_0 dV \right] dt
\end{aligned}
$$

where we have assumed a linear relationship between $\vec{B}$ and $\vec{H}$ in the last line. The surface integral was dropped since $\vec{E} \times \vec{H}$ due to local charges and currents decays atleast as fast as $1/r^4$. Now, α is a function of time alone and $\vec{B}_0$ is only a function of $\vec{r}$. The integrals separate out and we obtain

$$
W = -\int_0^T \alpha \frac{d\alpha}{dt} dt \int \frac{\left|\vec{B}_0\right|^2}{\mu_0} dV
$$

The time integral is

$$
\int_0^1 \alpha d\alpha = \frac{1}{2}
$$

Hence we get the result that the energy *gained* by the Magnetic Field is

$$
-W = \int \frac{\left|\vec{B}_0\right|^2}{2\mu_0} dV
$$

This particular derivation is only good for the static Ampere's Law. Otherwise, we get a generalized energy theorem that is called the Poynting Equation that includes radiation effects as well as energy in the Electric Field.

If there is no linear relationship between $\vec{B}$ and $\vec{H}$, we can still make some progress:

$$
\begin{aligned}
-W &= \int_0^T \left[\int \vec{H} \cdot \partial_t \vec{B} dV \right] dt \\
&= \int \left[\int_0^T \vec{H} \cdot \partial_t \vec{B} dt \right] dV \\
&= \int \left[\int_0^{B\text{max}} \vec{H} \cdot d\vec{B} \right] dV
\end{aligned}
$$

This is the familiar form of magnetic energy from your Electro-mechanical Energy Conversion course. The core of a ferromagnetic core is nonlinear and has hysterisis. Hence, $\vec{B}$ and $\vec{H}$ are not linearly related. However, you can still define a magnetic energy density

$$
\int_0^{B\text{max}} \vec{H} \cdot d\vec{B}
$$

If $\vec{H}$ cycles periodically, the energy put into the field per cycle is

$$
\oint \vec{H} \cdot d\vec{B}
$$

Since the fields become periodic after a few cycles, this energy does not go into the fields, but rather is dissipated as the heat of magnetisation.