Divergence of $\vec{E}$ related to ρ

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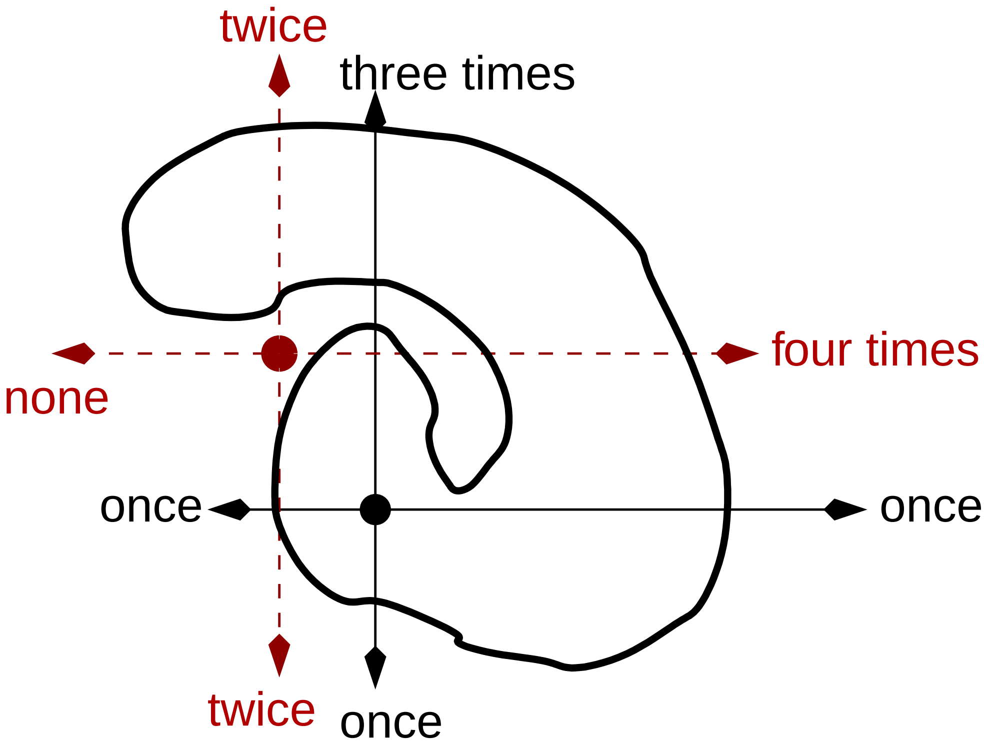

Consider an arbitrary closed volume. A charge lies inside the volume (black dot in figure). As can be seen, all rays from this charge pierce the surface an odd number of times, with outgoing rays numbering one more than entering rays. If we assign $+1$ to each outgoing ray and -1 to each entering ray, the rays will all yield a total of $+1$.

The red charge, on the other hand, lies outside the volume. As can be seen from the rays drawn, the rays pierce the suface an even number of times. An equal number of rays leave the surface as enter it. Thus, if we could "count" the rays from a charge, we have a perfect way to detect if the charge is within the surface or without. But how to count rays?

Let $4\pi N$ rays leave the charge. The solid angle available is 4π, hence N rays per square radian. Since the field due to a charge is along a constant direction, the rays represent the field lines. How many rays per unit area? The area of a sphere of radius r is $4\pi r^2$. So we have N/r^2 rays per unit area, i.e., the *density of rays* is proportional to (or a measure of) the Electric Field itself. All we have to do is to count the Electric field along the surface of any volume and we know the enclosed charge!

But how to count the Field? The surface is not a sphere. It is arbitrary and at most points $\vec{E}$ is not normal to it. What is the solid angle presented by the surface to the charge? The answer is obvious:

$$d\Omega = \hat{E} \cdot \vec{dS}$$

Thus, the total number of field lines is given by

$$\oint \left|\vec{E}\right| d\Omega = \oint \vec{E} \cdot \vec{dS} = \begin{cases} 0, & \text{charge outside volume} \\ Q/\varepsilon_0, & \text{charge inside volume} \end{cases}$$

When we take this result to the limit of small volumes, and use the divergence theorem we obtain

$$\int \nabla \cdot \vec{E} dV = \oint \vec{E} \cdot \vec{dS} = \int \frac{\rho}{\varepsilon_0} dV$$

i.e.,

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$