The Material Response to Applied $\vec{B}$ and the Magnetic Field Intensity $\vec{H}$

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This discussion parallels the discussion of the dielectric response. So we will go a little quickly, assuming that you have already read that writeup.

If we have a set of known currents, the field due to those currents is given by

$$\nabla \times \vec{B} = \mu_0 \vec{j} \tag{1}$$

This equation is true always, provided we could specify all the currents, including those inside materials. Our problem lies in the fact that we do not know the detailed location and orientation of each of the atoms inside materials. We need a simpler approach that makes intelligent assumptions about the things we don't know anyway. Once again, there are two kinds of materials, namely conductors and insulators.

Conductors

There are important properties of conductors that complicates magnetism.

- **Normal Conductors:** We augment Eq. 1 by adding boundary conditions and by generalizing Ohm's Law. Suppose we had some current carrying species s (for example, electrons and holes in a semiconductor). Then,

$$\begin{aligned} \nabla \cdot \vec{j}_s &= 0 \\ \vec{j}_s &= \sigma_s \left(\vec{E} + \vec{v}_s \times \vec{B} \right) \\ \nabla \times \vec{B} &= \mu_0 \sum_s \vec{j}_s \\ \vec{j}_s &= n_s q_s \vec{v}_s \end{aligned}$$

  This problem is extremely complex to solve, unless we can make simplifying assumptions:

  - Self-magnetic field is negligible, two species ... semi-conductor devices, Hall effect.
  - Electrons dominate the current, positively charged atoms dominate the mass, reduced equations for single species ... magnetohydrodynamics, used to describe solar structure, the corona, and the solar wind.
  - Electrons dominate the current, but atoms neutral in the bulk. Mechanical energy dominates Electromagnetic energy ... the dynamo effect and the interior of the Earth. The origin of planetary magnetic fields.

  - Electrons and one or more ion species present; each is thermal, but the species are at different temperatures ... plasma physics in many experimental systems, plasma physics of accretion disks of black holes.
  - Suspended dust becomes charged and begins to participate in the dynamics ... dusty and colloidal plasmas, important in plasma chemistry.

- **Ferromagnetic Conductors:** Certain conductors have all the properties above, and in addition have important magnetic properties. Iron is the most famous example. We will only touch upon these materials in this course.

Insulators

The second type of material is the insulator. Here, there are no currents flowing. How does a magnetic field affect such a material and how does the field get modified by the material response?

When an atom is in isolation, it has electrons in various quantum levels. Each electron has some spin, and quantum levels come in pairs that correspond to the sense of the spin of the electron in that state. In the absence of a magnetic field, these two spin states are degenerate and correspond to the same energy. However, in the presence of a magnetic field, the states split in energy and the wave functions themselves are slightly distorted. The result is that the atom goes from having no intrinsic circulating currents to having an induced current loop.

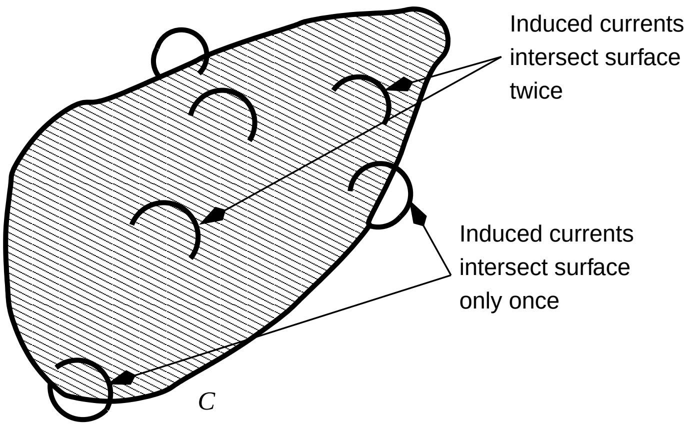
Each atom is different, and its induced currents are in different planes that depend on the interactions between the atom and its neighbours. Thus, the $\vec{m}$ of each atom points in arbitrary directions. But as I argued in connection with dielectrics, we can talk about an average response of the material to an applied magnetic field $\vec{B}$. Once again it is a complicated, nonlinear function of the applied field, but let us look at the regime where it is linear.

$$m_i = \langle \alpha_{ij} \rangle B_j$$

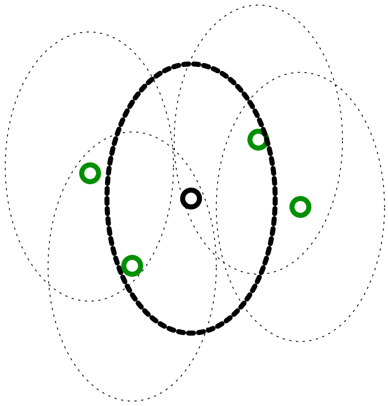
$\vec{m}$ can be thought of as a current loop, with its direction pointing out of the plane of the loop (following the right-hand rule). Its magnitude is IA where I is the miniature current of the electrons and A is the area cross-section of the loop. For a given "diamagnetic" material, $\langle \alpha \rangle$ can be experimentally determined.

Let us now consider how this induced current affects Ampere's Law. Ampere's Law states

$$\oint_C \vec{B} \cdot \vec{dl} = \mu_0 I_{\text{encl}} = \mu_0 \int_{S_C} \vec{j}_{\text{real}} \cdot \vec{dS} + \mu_0 \int_{S_C} \vec{j}_{\text{ind}} \cdot \vec{dS}$$



Induced current loops that intersect the Stoke's surface at two points do not contribute to net current. Only induced current loops that intersect the surface at one point contribute to net current. Thus the induced current comes from a layer near the edge of the surface. But what is the contribution of that layer?



The picture above shows the situation as it would look as we move along the curve C. The $\vec{m}$ direction would not be aligned along $\vec{dl}$ and so the current loops would look like ellipses. The green circles are the centres of the ellipses. It is clear from the picture that those and only those ellipses whose centres lie within the dark ellipse contribute to induced current. Ellipses whose centres are to the left have cancelling contributions, while ellipses whose centres are to the right do not contribute at all.

The volume of atoms contributing to this current is therefore equal to

$$\text{Area of ellipse} \times \text{length of curve}$$

Hence the induced current integral becomes

$$\oint_C nIA\hat{m} \cdot \vec{dl} = \oint n\vec{m} \cdot \vec{dl}$$

Ampere's Law therefore becomes

$$\oint_C \left(\frac{\vec{B}}{\mu_0} - n\vec{m} \right) \cdot \vec{dl} = \int_{S_C} \vec{j}_{\text{true}} \cdot \vec{dS}$$

Once again, using $\vec{m} = \langle \alpha \rangle \vec{B}$ we obtain a new quantity

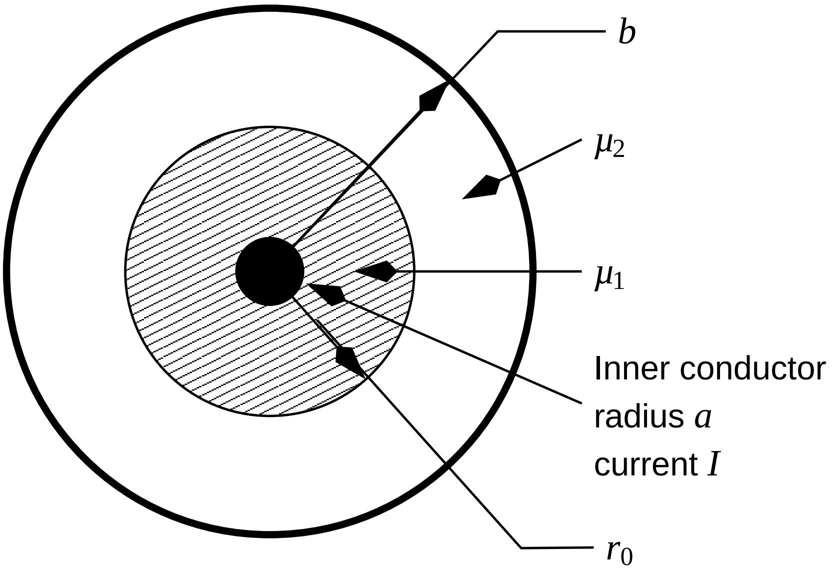
$$\vec{H} = \left(\frac{1}{\mu_0} - n \langle \alpha \rangle \right) \vec{B}$$

In terms of $\vec{H}$, Ampere's Law becomes

$$\oint_C \vec{H} \cdot \vec{dl} = \int_{S_C} \vec{j}_{\text{true}} \cdot \vec{dS}$$

An Example

Consider a coaxial cable, with two layers of permeable material.



The inner cable carries a current I out of the page, in the z direction. The problem is to determine the resulting $\vec{B}$ field.

We cannot use ...

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{\text{encl}}$$

since that will involve determining the extent of the induced current. However, we do know that the symmetry of the problem says that

$$\vec{B} = \vec{B}(r)$$

and the divergence theorem says that $B_r = 0$.

What comes to our rescue is

$$\vec{B} = \mu \vec{H}$$

What this means is that there is another quantity $\vec{H}$ that is collinear with (points in the same direction as) $\vec{B}$, but whose equation is simpler. Since $\vec{B}$ depends only on r and has no r component, so does $\vec{H}$. **Note:** We cannot use $\nabla \cdot \vec{H} = 0$. It is not even true in general.

We can now use Ampere's law now to determine H_θ and H_z. Using a circular loop in the x-y plane centred about the wire

$$\oint \vec{H} \cdot \vec{dl} = I$$

which implies

$$H_\theta = \frac{I}{2\pi r}$$

Using a loop with sides along z and r, we can determine H_z. But that requires a θ-direction current, which is not present. Hence $H_z = 0$, and

$$\vec{H} = \frac{I}{2\pi r}\hat{\theta}$$

This immediately yields the simple result

$$\vec{B} = \frac{\mu(r)}{2\pi r}\hat{\theta}$$