Far Field of an Arbitrary Current Loop

29th January 2007

We have obtained the magnetic field far from a circular current loop (with quite a bit of work) in the *Far field due to a current loop* entry on the site. Here we show that the result is actually completely general to any current loop of any shape, i.e., for *any* loop,

$$A(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}$$

where $\vec{m} = \int \vec{r}' \times \vec{j}(\vec{r}')dV'/2$.

First, we need (guess?!) two vector identities. First, an obvious result:

$$\begin{aligned}
\int \vec{j}(\vec{r}')dV' &= \frac{1}{3}\int \vec{j}(\vec{r}')\nabla'\cdot\vec{r}'dV' \\
&= \frac{1}{3}\int \nabla'\cdot\left(\vec{j}(\vec{r}')\vec{r}'\right)dV' - \frac{1}{3}\int \vec{r}'\nabla'\cdot\vec{j}(\vec{r}')dV' \\
&= \oint \vec{j}(\vec{r}')\vec{r}'\cdot d\vec{S} = 0
\end{aligned}$$

provided $\vec{j}$ is a localized current source. This merely says that since charge cannot be created or destroyed, current must flow in loops and must add up vectorially to zero.

The second identity is peculiar, but very important.

$$\begin{aligned}
\vec{r}\times\int \vec{r}'\times\vec{j}(\vec{r}')dV' &= \varepsilon_{ikl}\hat{x}_i x_k \int \varepsilon_{lmn} x'_m j_n(\vec{r}')dV' \\
&= \hat{x}_i x_k \int x'_m j_n dV' \left(\delta_{im}\delta_{kn} - \delta_{in}\delta_{km}\right) \\
&= \hat{x}_i x_k \int \left(x'_i j_k(\vec{r}') - x'_k j_i(\vec{r}')\right)dV'
\end{aligned}$$

Now

$$\begin{aligned}
\int x'_i x'_k \nabla'\cdot\vec{j}(\vec{r}')dV' &= 0 \\
&= -\int \vec{j}(\vec{r}')\cdot\nabla\left(x'_i x'_k\right) \\
&= -\int \left(j_i(\vec{r}')x'_k + j_k(\vec{r}')x'_i\right)
\end{aligned}$$

which implies

$$\int \left(x'_i j_k - x'_k j_i\right)dV' = -2\int x'_k j_i dV'$$

Then

$$
\begin{aligned}
\vec{r} \times \int \vec{r}' \times \vec{j}(\vec{r}')dV' &= \hat{x}_i x_k \int \left(x_i' j_k(\vec{r}') - x_k' j_i(\vec{r}') \right) dV' \\
&= -2x_k \int x_k' \vec{j}(\vec{r}')dV' \\
&= -2 \int \left(\vec{r} \cdot \vec{r}' \right) \vec{j}(\vec{r}')dV'
\end{aligned}
$$

Now we are ready to proceed with the result:

$$
\begin{aligned}
\vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi} \int \frac{\vec{j}(\vec{r}')}{R_{12}} dV' \\
&= \frac{\mu_0}{4\pi} \int \frac{\vec{j}(\vec{r}')}{r} \left(1 + \frac{\vec{r} \cdot \vec{r}'}{r^2} + \dots \right) dV' \\
&= -\frac{\mu_0}{4\pi} \frac{\vec{r}}{2r^3} \times \int \vec{r}' \times \vec{j}(\vec{r}')dV' \\
&= \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}
\end{aligned}
$$

where

$$
\vec{m} = \frac{1}{2} \int \vec{r}' \times \vec{j}(\vec{r}')dV'
$$

For the special case of a circular loop, this becomes

$$
\vec{m} = \frac{1}{2} \int_0^{2\pi} a\hat{r} \times I\hat{\theta} a d\theta = \pi a^2 I \hat{z}
$$

Note that in this new derivation, the loop does not need to be in a plane. Indeed, it does not need to be a loop and can be a distributed current. All it requires is that $\vec{j}$ be confined to a local region, so that the surface integral at infinity can be discarded. So a finite solenoid looks like a dipole at far distances. A U-shaped magnet's field looks like a dipole, etc.