Far Field due to a Current Loop

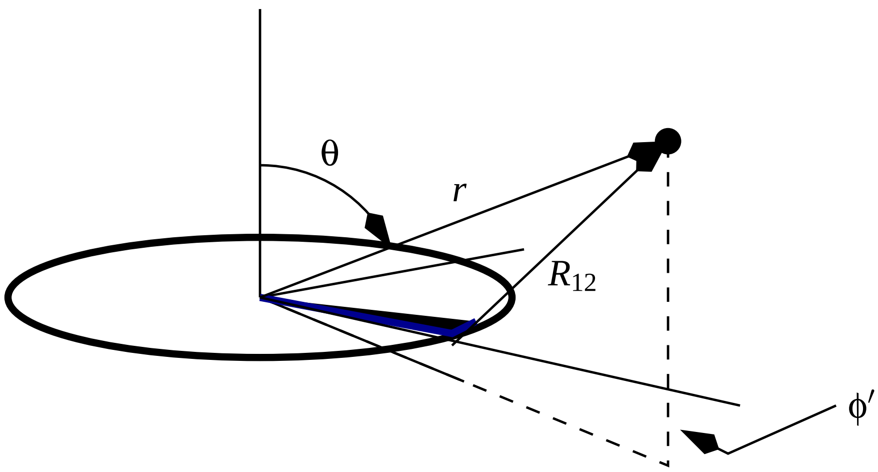
24th January 2007

This derivation follows section 2.10 of the textbook. Consider a current loop of radius a, centered at the origin and carrying a current I_0.

Looking back to the solenoid calculation, we obtain the vector potential due to a loop as

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} I_0 \int_{-\pi}^{\pi} \frac{\cos\phi'}{R_{12}} a\,d\phi' \qquad (1)$$

The problem is to evaluate Eq. 1 when $R_{12} \gg a$.



The best coordinate system to tackle this problem is the spherical polar coordinate system (r, θ, ϕ). The integral involves R_{12} which we must express in terms of r, θ, ϕ and ϕ'. We rotate the system till $\phi = 0$.

Let ψ be the angle between $\vec{r}$ and $\vec{r}'$. then

$$R_{12} = \sqrt{r^2 + a^2 - 2ra\cos\psi}$$

In the (r, θ, ϕ) coordinate system, the vector $\vec{r}'$ is given by

$$\begin{aligned} \vec{r}' &= a\cos\phi'\hat{x} + a\sin\phi'\hat{y} \\ &= a\cos\phi'\left(\hat{r}\sin\theta + \hat{\theta}\cos\theta\right) + a\sin\phi'\hat{\phi} \end{aligned}$$

Thus,

$$ar\cos\psi = \vec{r}\cdot\vec{r}' = ra\cos\phi'\sin\theta$$

This allows us to express R_{12} in terms of known quantities

$$R_{12} = \sqrt{r^2 + a^2 - 2ar\cos\phi' \sin\theta} \tag{2}$$

Taylor expanding Eq. 2, Eq. 1 becomes

$$A_\phi = \frac{\mu_0}{4\pi} Ia \int_{-\pi}^{\pi} \cos\left(\phi'\right) \left(\frac{1 + a\sin\theta\cos\phi'/r}{r} \right) d\phi'$$

Only the $\cos^2\phi'$ term survives the integration and we obtain

$$A_\phi = \frac{\mu_0}{4\pi} I \left(\pi a^2\right) \frac{\sin\theta}{r^2} = \frac{\mu_0}{4\pi} m \frac{\sin\theta}{r^2} \tag{3}$$

where m is called the magnetic dipole moment of the loop. We take the curl of $\vec{A}$ to get

$$\begin{aligned}
\vec{B} &= \frac{1}{r^2\sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta} \left[\hat{r} r \partial_\theta \left(\sin\theta A_\phi\right) - r\sin\theta\hat{\theta}\partial_r \left(rA_\phi\right) \right] \\
&= \frac{\hat{r}}{r\sin\theta} \partial_\theta \left(\sin\theta A_\phi\right) - \frac{\hat{\theta}}{r} \partial_r \left(rA_\phi\right)
\end{aligned}$$

Using Eq. 3 this becomes

$$\vec{B} = \frac{\mu_0}{4\pi} m \left(\frac{2\cos\theta\hat{r}}{r^3} + \frac{\sin\theta\hat{\theta}}{r^3} \right) \tag{4}$$

This has exactly the same structure as the answer we get for the Electric Dipole:

$$\phi = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{1}{4\pi\varepsilon_0} \frac{p\cos\theta}{r^2}$$

which gives for the Electric Field,

$$\vec{E} = -\hat{r}\partial_r\phi - \frac{\hat{\theta}}{r}\partial_\theta\phi \tag{5}$$

$$= \frac{1}{4\pi\varepsilon_0} p \left(\frac{2\cos\theta\hat{r}}{r^3} + \frac{\sin\theta\hat{\theta}}{r^3} \right) \tag{6}$$